What is spectrum?

Continuum + Spectral Features

absorption / emission

Blackbody

free-free

atom
molecules

high res.

ice
dust

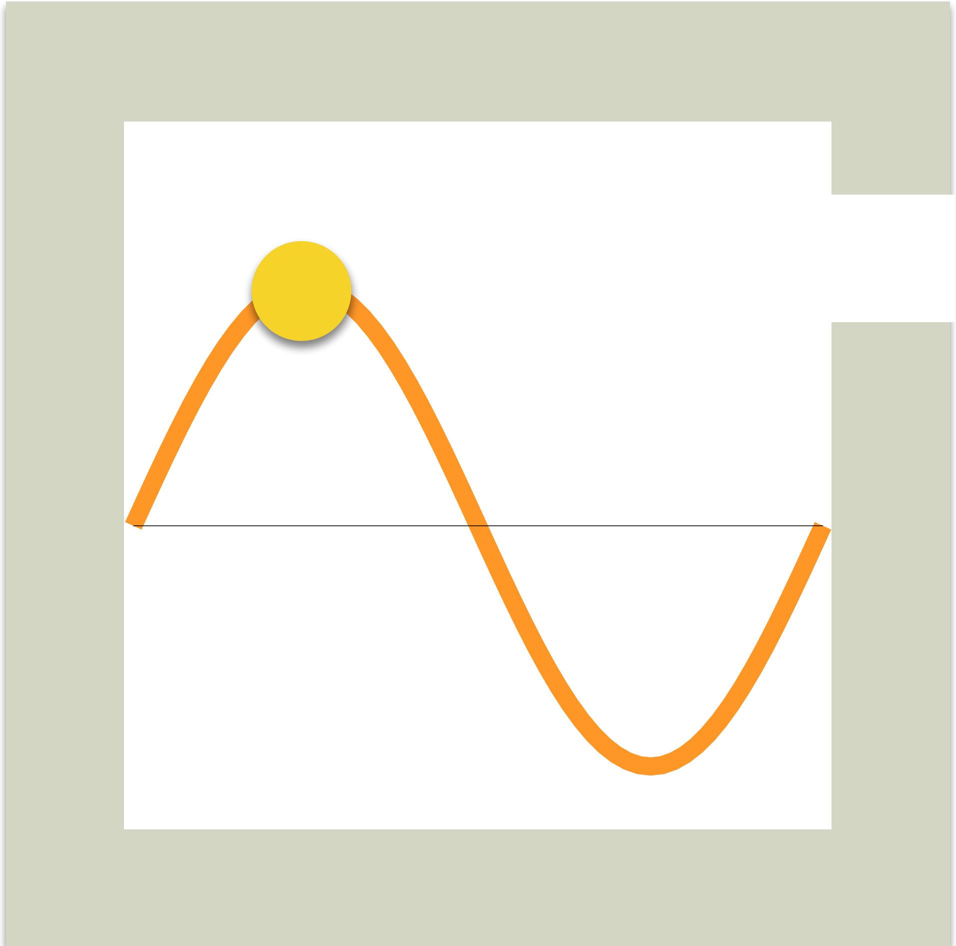
low res.

1 blackbody
2 radiation
3 Thomson scattering
4 radiation transfer
5 Einstein coefficients

Blackbody from scratch

The probability that a system is in the energy level ε_s is proportional to

$$p \propto \exp\left(-\frac{\varepsilon_s}{\tau}\right)$$



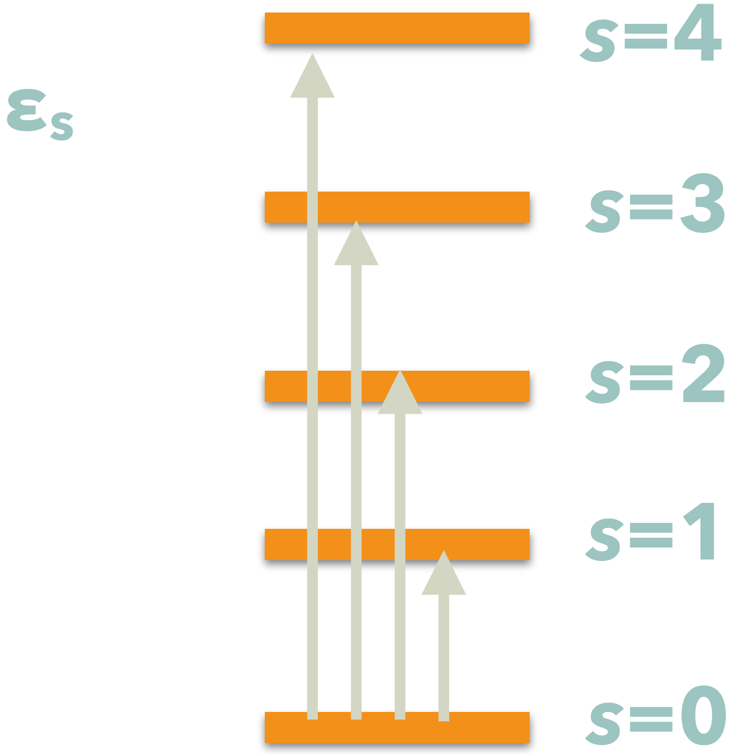
system photons in a cavity

mode ω $(=2\pi\nu)$

energy $\hbar\omega$

temperature τ $(=kT)$

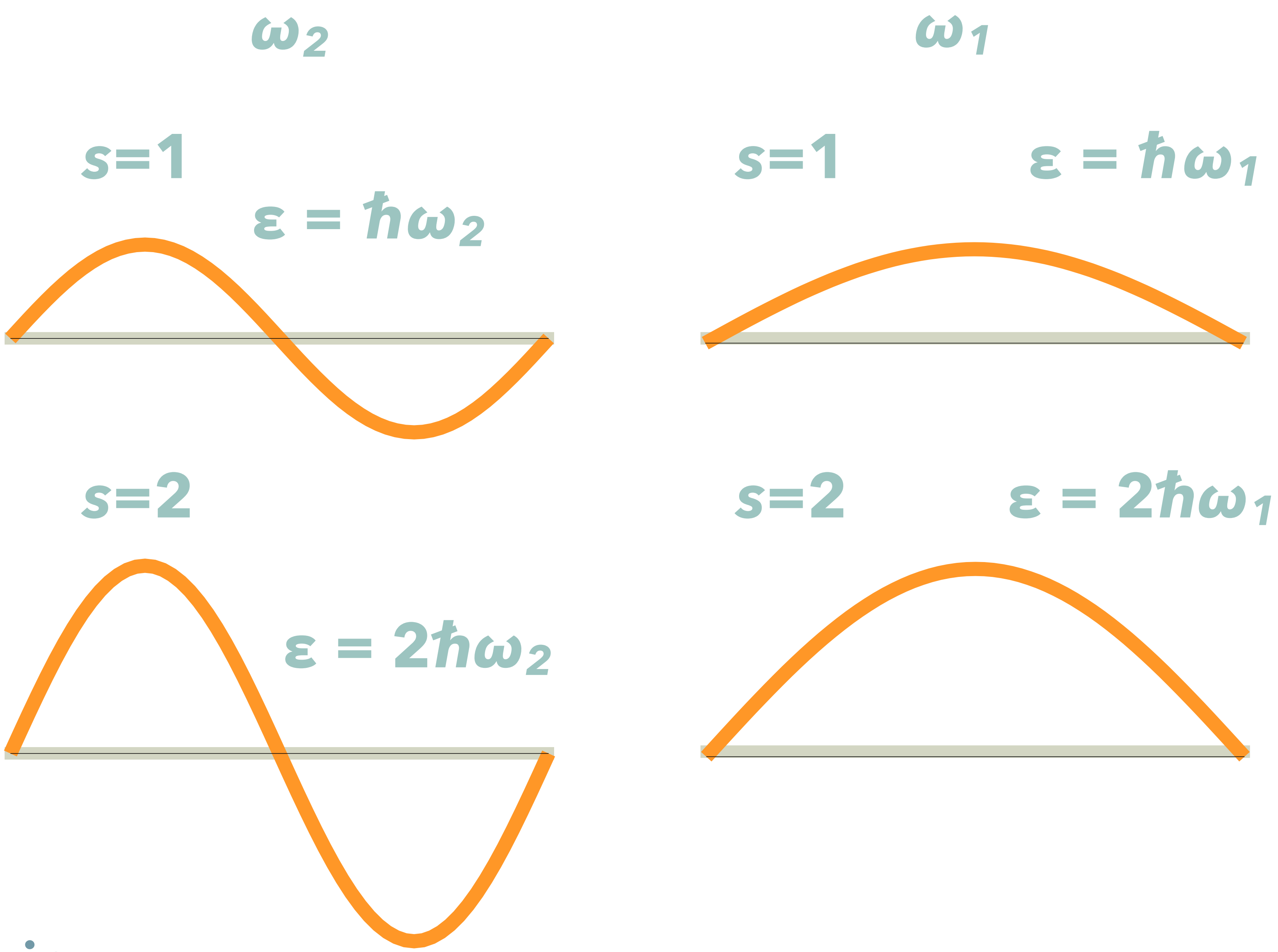
1. how many photons are in the mode ω?
2. how many modes are in whole space?
   (phase space)



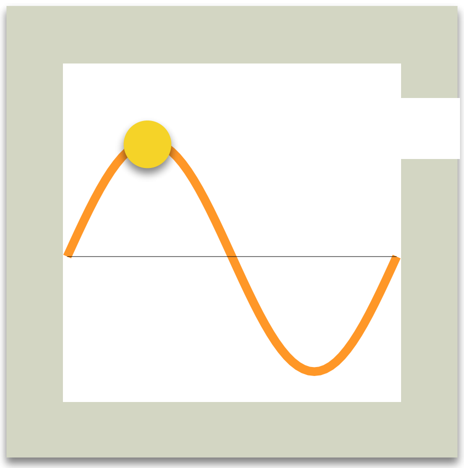
$\varepsilon_s = s\hbar\omega_1$

$p \propto \exp(-\frac{\varepsilon_s}{\tau})$

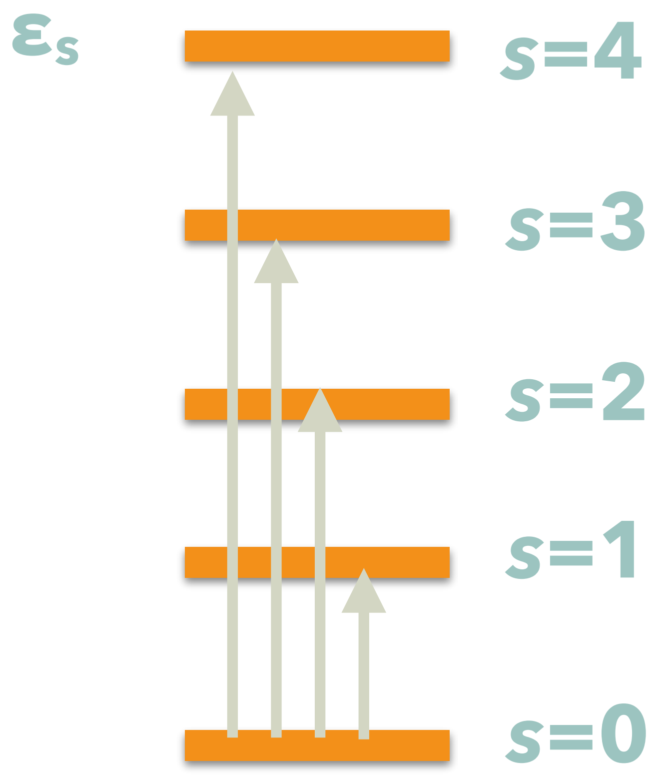
talking about a single mode currently



s photon in ω is equivalent to an oscillator being in s-th orbital



how many photons are in the mode ω?



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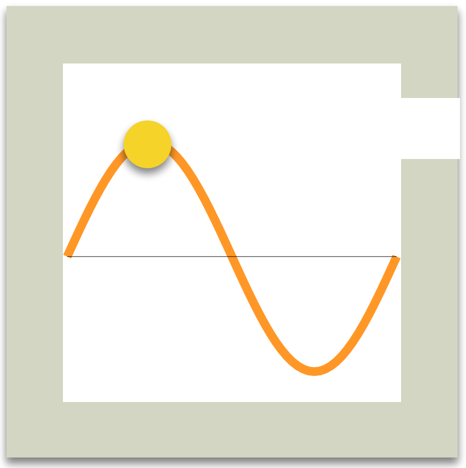


1

how many photons are in the mode ω?

partition function

(sum of all possibilities)

$$p \propto \exp(-\frac{\varepsilon_s}{\tau})$$

$$\varepsilon_s = s\hbar\omega$$

$$Z=\sum_s \exp(-\frac{\varepsilon_s}{\tau})$$

$$=\sum_s \exp(-\frac{s\hbar\omega}{\tau})$$

this is a geometric progression

$$Z=\frac{1}{1-\exp(-\frac{\hbar\omega}{\tau})}$$

$$\sum_{i=1}^{n} x^{i-1}=\frac{1-x^n}{1-x}$$

$$x < 1$$

$$\frac{1}{1-x}$$

average occupation of ω

$$<s> = \frac{\sum_s s\exp(-\frac{s\hbar\omega}{\tau})}{Z}$$

photons

average occupation of ω

$$\langle s \rangle = \frac{\sum_{s} s \exp(- \frac{s\hbar\omega}{\tau})}{Z}$$

$$Z = \frac{1}{1- \exp(- \frac{\hbar\omega}{\tau})}$$

$$y = \frac{\hbar\omega}{\tau}$$

$$\sum_{s} s \exp(-sy) = - \frac{d}{dy} \sum_{s} \exp(-sy)$$

$$= \frac{d}{dy} \frac{-1}{1 - \exp(-y)} = \frac{\exp(-y)}{[1 - \exp(-y)]^2}$$

$$\langle s \rangle = \frac{\exp(-y)}{1 - \exp(-y)} = \frac{1}{\exp(y) -1}$$

$$\sum_{i=1}^{n} x^{i-1} = \frac{1-x^n}{1-x}$$

$$x < 1$$

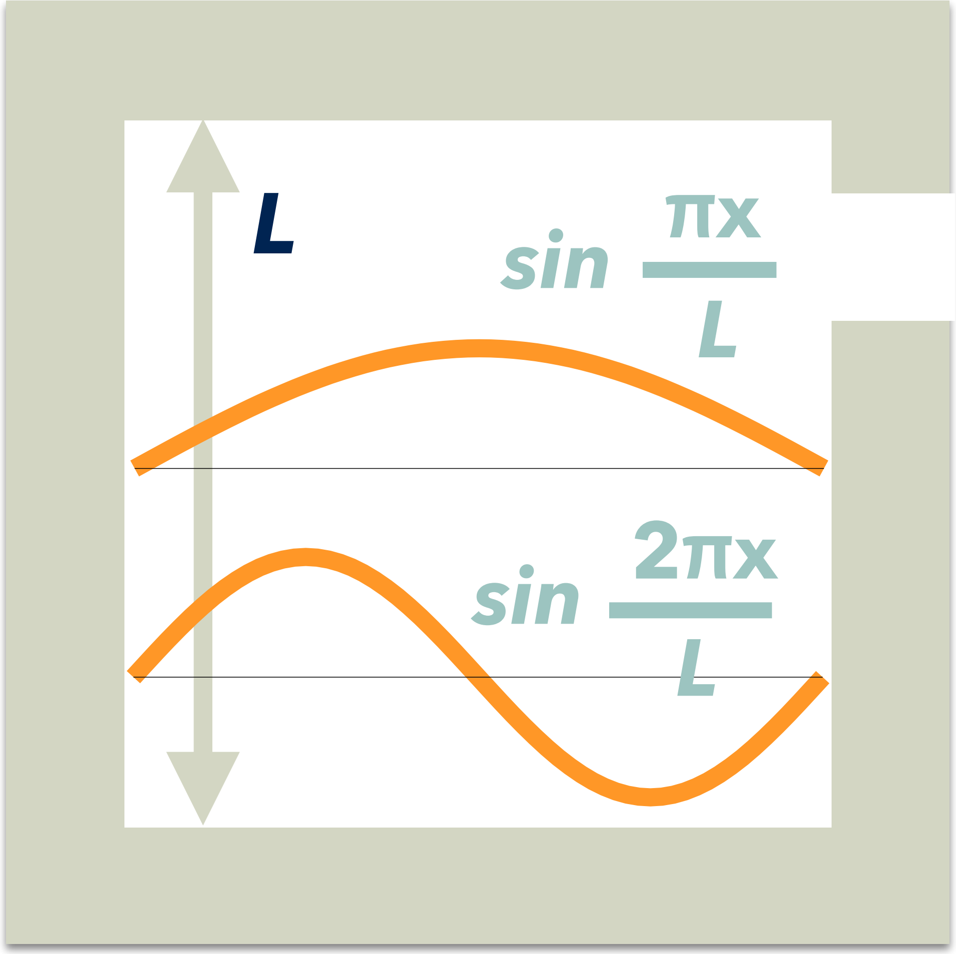
$$\frac{1}{1-x}$$

$$\langle \varepsilon \rangle = \frac{\hbar\omega}{\exp(\frac{\hbar\omega}{\tau}) -1}$$

average energy in mode ω

2 how many modes are in phase space

standing wave



wave equation

$$\frac{\partial^2 E}{\partial t^2} = c^2 \frac{\partial^2 E}{\partial x^2}$$

$$\mathbf{n} = (n_x, n_y, n_z)$$

propergation of wave

$$\mathbf{E} = (E_x, E_y, E_z)$$

$$E_x = E_{x0}\,\underbrace{\sin \omega t}_{\textbf{temporal}}\,\cos\frac{n_x \pi x}{L}\,\sin\frac{n_y \pi y}{L}\,\sin\frac{n_z \pi z}{L}$$

$$E_y = E_{y0}\sin \omega t\,\sin\frac{n_x \pi x}{L}\,\cos\frac{n_y \pi y}{L}\,\sin\frac{n_z \pi z}{L}$$

$$E_z = E_{z0}\sin \omega t\,\sin\frac{n_x \pi x}{L}\,\sin\frac{n_y \pi y}{L}\,\cos\frac{n_z \pi z}{L}$$

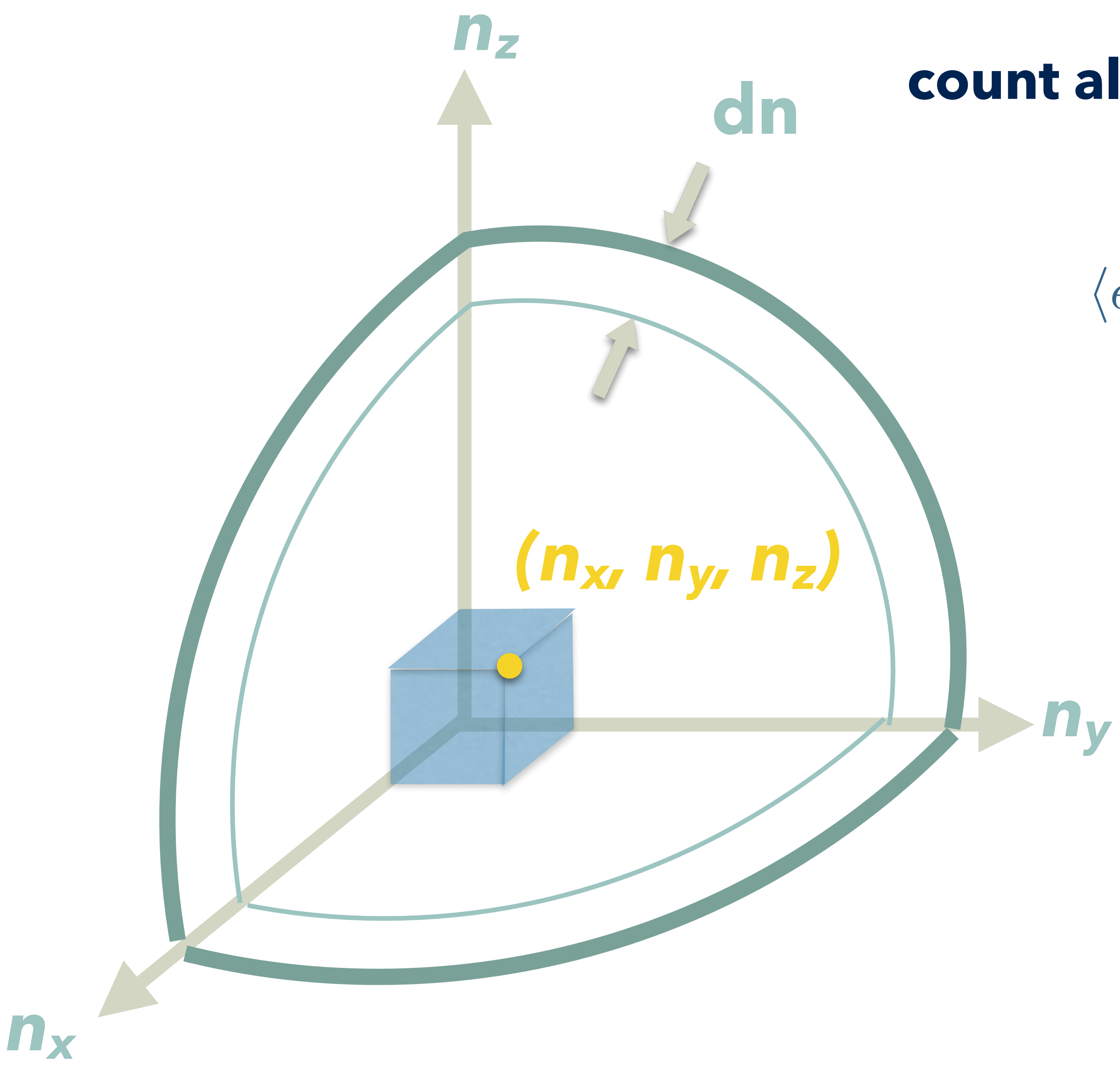
$$\omega^2 = c^2\left[\left(\frac{n_x \pi}{L}\right)^2 + \left(\frac{n_y \pi}{L}\right)^2 + \left(\frac{n_z \pi}{L}\right)^2\right]$$

$$n^2 = n_x^2 + n_y^2 + n_z^2$$

$$\omega = \frac{n\pi c}{L}$$ **mode**

we already know how the mode is occupied

count all modes in phase space



$$\omega = \frac{n\pi c}{L}$$

$$n = \frac{\omega L}{\pi c}$$

$$\langle \epsilon \rangle = \frac{\hbar\omega}{\exp\frac{\hbar\omega}{\tau} - 1}$$

positive polarization

$$U = 4\pi n^2 \int_0^\infty <\varepsilon> dn \times \frac{1}{8} \times 2$$

$$U = 4\pi n^2 \int_0^\infty \frac{\hbar\omega}{\exp\frac{\hbar\omega}{\tau} - 1} dn \times \frac{1}{8} \times 2$$

$$= \pi \left(\frac{L}{\pi c}\right)^3 \int_0^\infty \frac{\hbar\omega^3}{\exp\frac{\hbar\omega}{\tau} - 1} d\omega$$

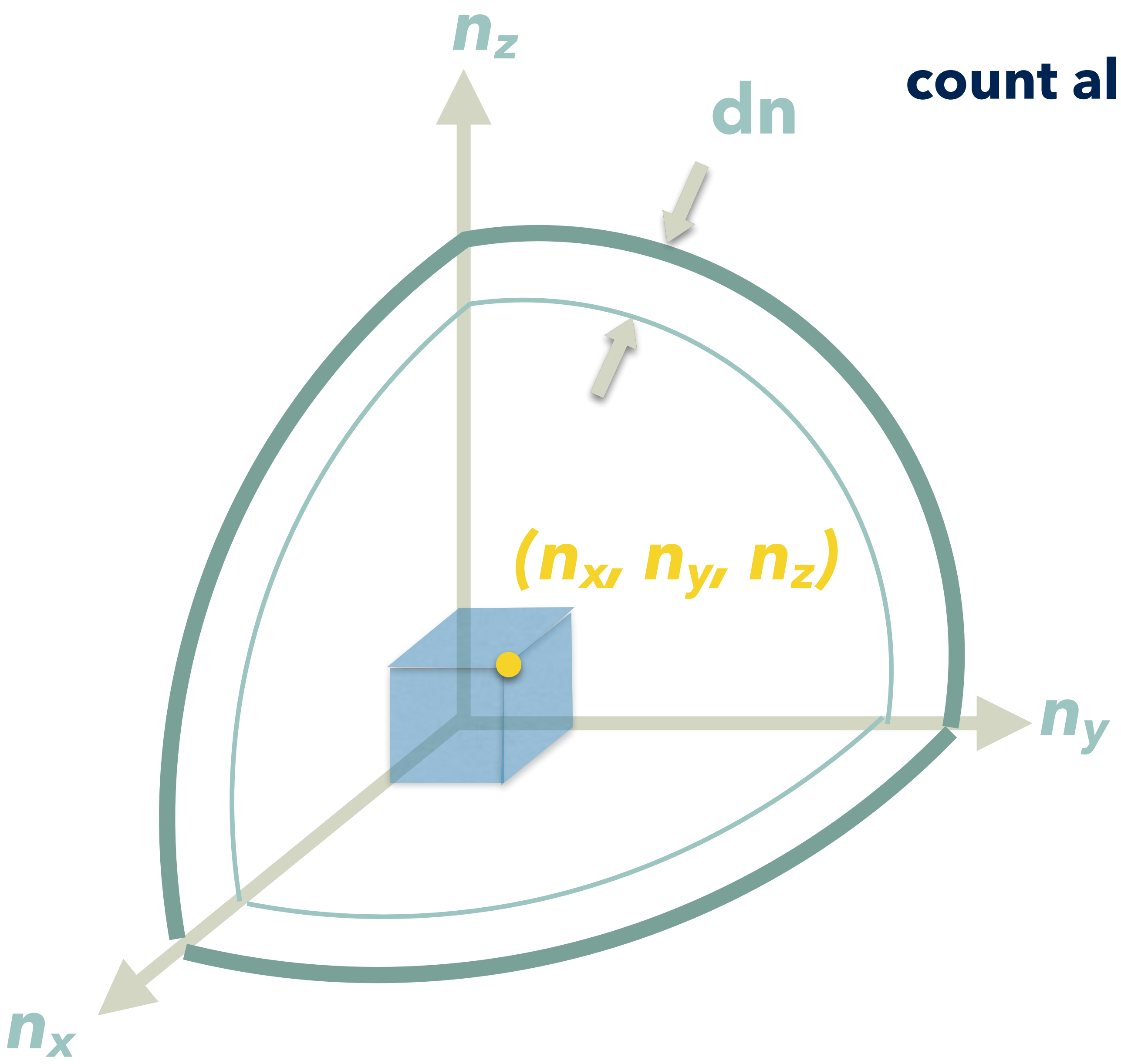
$$u(\omega)d\omega = \frac{\pi}{\pi^3 c^3} \frac{\hbar\omega^3}{\exp\frac{\hbar\omega}{\tau} - 1} d\omega$$

$$= \frac{1}{\pi^2 c^3} \frac{\hbar\omega^3}{\exp\frac{\hbar\omega}{\tau} - 1} d\omega$$

we do not integrate

$$u = \frac{U}{V} = \frac{U}{L^3}$$

count all modes in phase space



$$\omega = \frac{n\pi c}{L}$$

ω=2πν

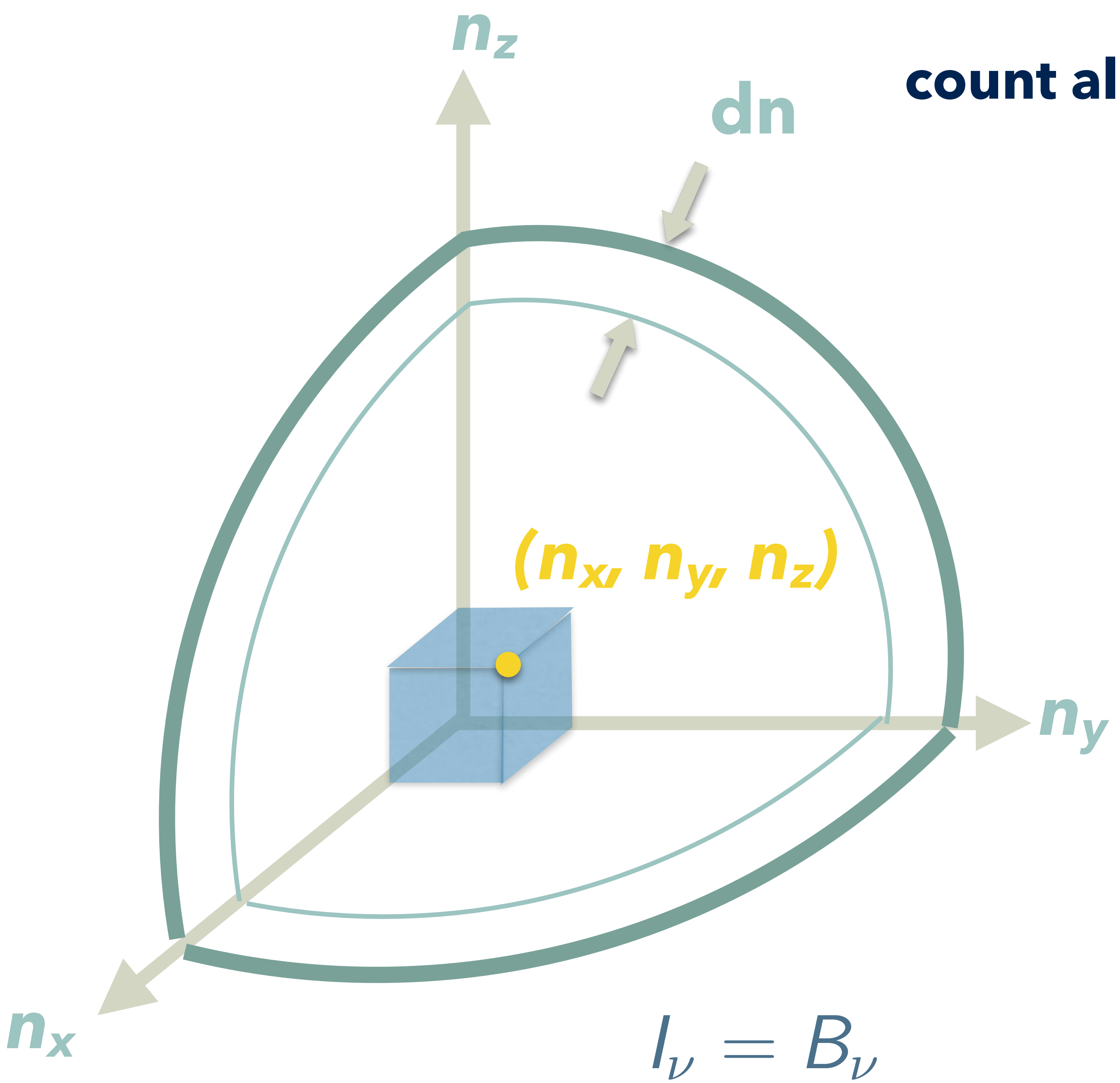
$$u(\nu) = u(\omega) \cdot 2\pi$$

$$= \frac{1}{\pi^2 c^3} \frac{h(2\pi)^3 \nu^3}{\exp \frac{h\nu}{kT} - 1}$$

$$= \frac{8\pi}{c^3} \frac{h\nu^3}{\exp \frac{h\nu}{kT} - 1}$$

$$u(\omega) d\omega = \frac{\pi}{\pi^3 c^3} \frac{\hbar \omega^3}{\exp \frac{\hbar\omega}{\tau} - 1} d\omega$$

$$= \frac{1}{\pi^2 c^3} \frac{\hbar \omega^3}{\exp \frac{\hbar\omega}{\tau} - 1} d\omega$$



count all modes in phase space

$$\omega = \frac{n\pi c}{L}$$

ω=2πν

$$u(\nu) = u(\omega) \cdot 2\pi$$

$$= \frac{1}{\pi^2 c^3} \frac{h(2\pi)^3 \nu^3}{\exp \dfrac{h\nu}{kT} - 1}$$

$$= \frac{8\pi}{c^3} \frac{h\nu^3}{\exp \dfrac{h\nu}{kT} - 1}$$

$$I_\nu = B_\nu$$

$$B(\nu) = \frac{c}{4\pi} u(\nu)$$

$$= \frac{c}{4\pi} \frac{8\pi}{c^3} \frac{h\nu^3}{\exp \dfrac{h\nu}{kT} - 1}$$

$$= \frac{2h\nu^3}{c^2} \frac{1}{\exp \dfrac{h\nu}{kT} - 1}$$

Stefan-Boltzmann law

total energy from a unit surface?

$$B(\nu) = \frac{2h\nu^3}{c^2}\frac{1}{\exp\dfrac{h\nu}{kT} - 1}$$

Planck function

$$I_\nu = B_\nu$$

$$F_\nu = \pi I_\nu$$

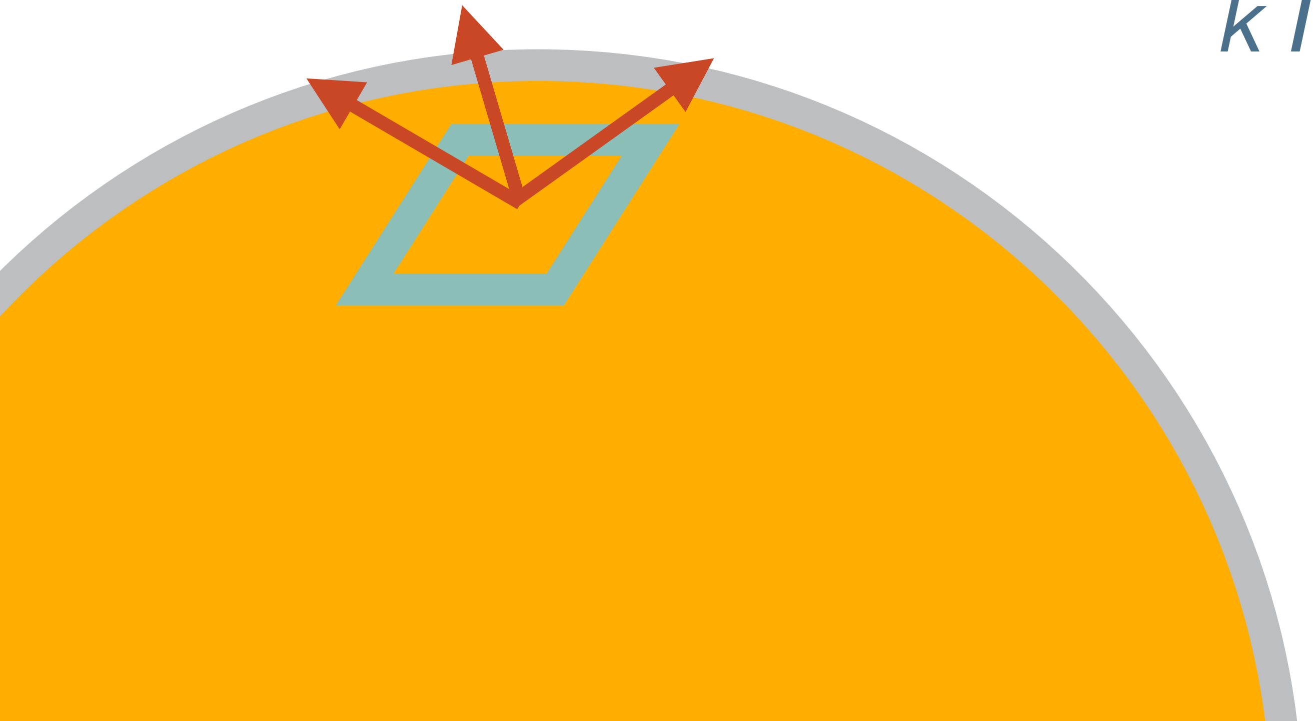
$$= \pi B_\nu$$

$$E = \int_0^\infty B_\nu d\nu$$

$$F = \sigma T^4$$

$$u = \frac{h\nu}{kT}$$

$$\int_0^\infty \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$



1 **Stefan-Boltzmann constant**

$$\sigma = \frac{2\pi^5 k^4}{15c^2 h^3}$$

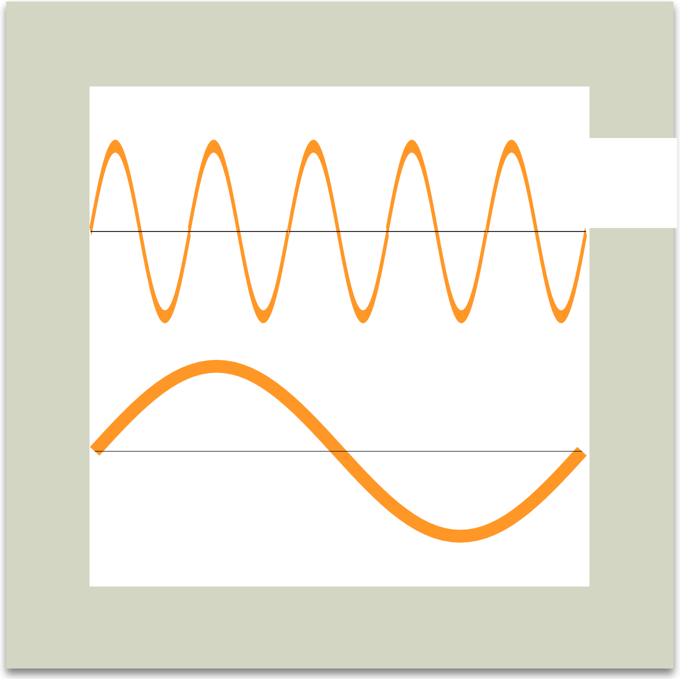
Blackbody is

① optically thick

if not, call it a graybody

blackbody is not a shape of spectrum
it is an absolute level

② strongest emission at given frequency
at given temperature



all possible modes are fully occupied
cannot put photons any more

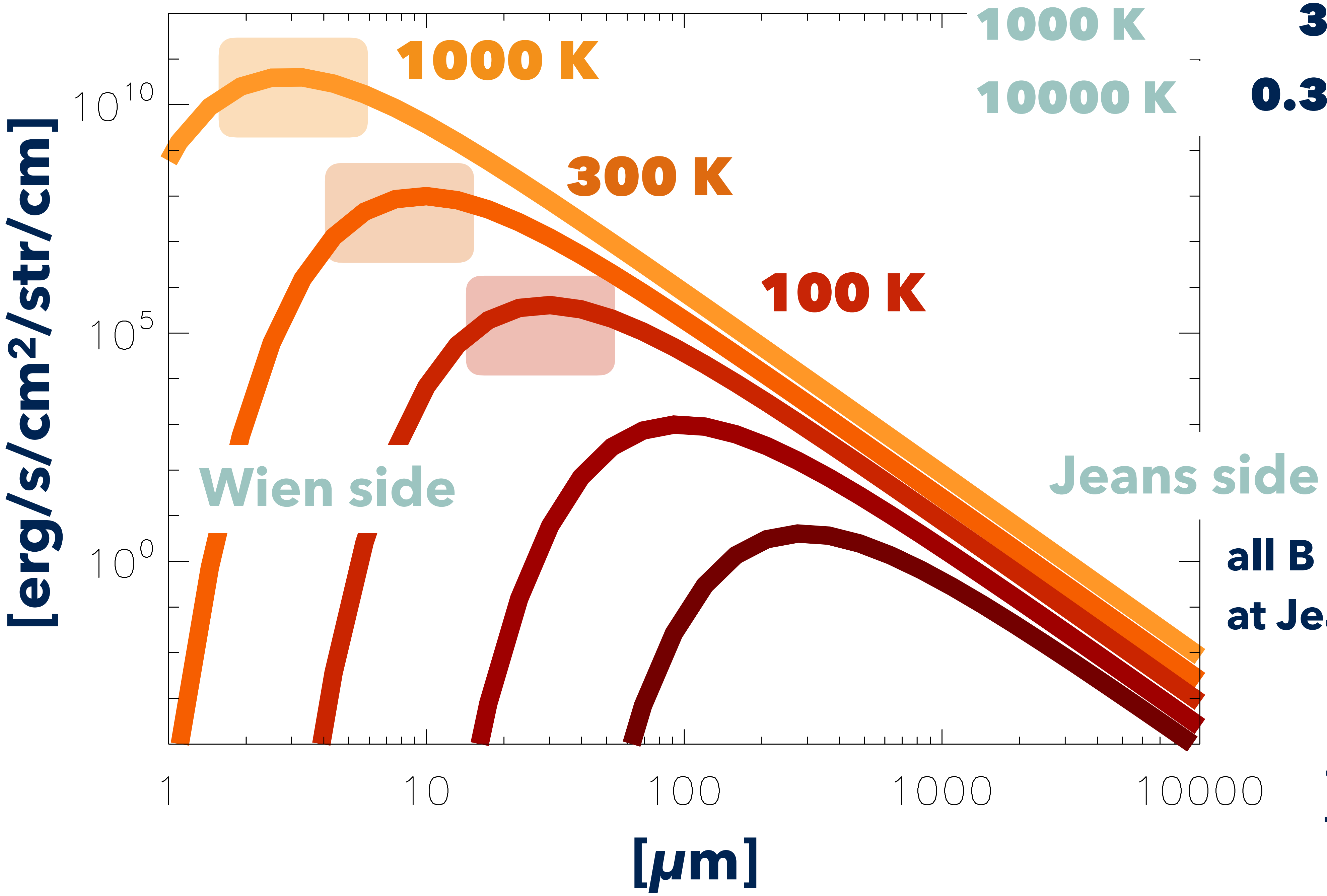
whatever the mechanism is,
when you make an emission
stronger and stronger,
and reached *B(T)*,
that is the end.

Planck function

you only see the peak of Blackbody

$\lambda T = 3000 \; [\mu m \cdot K]$

10 K	300 μm	sub-mm
100 K	30 μm	mid-IR
1000 K	3 μm	near-IR
10000 K	0.3 μm	UV optical

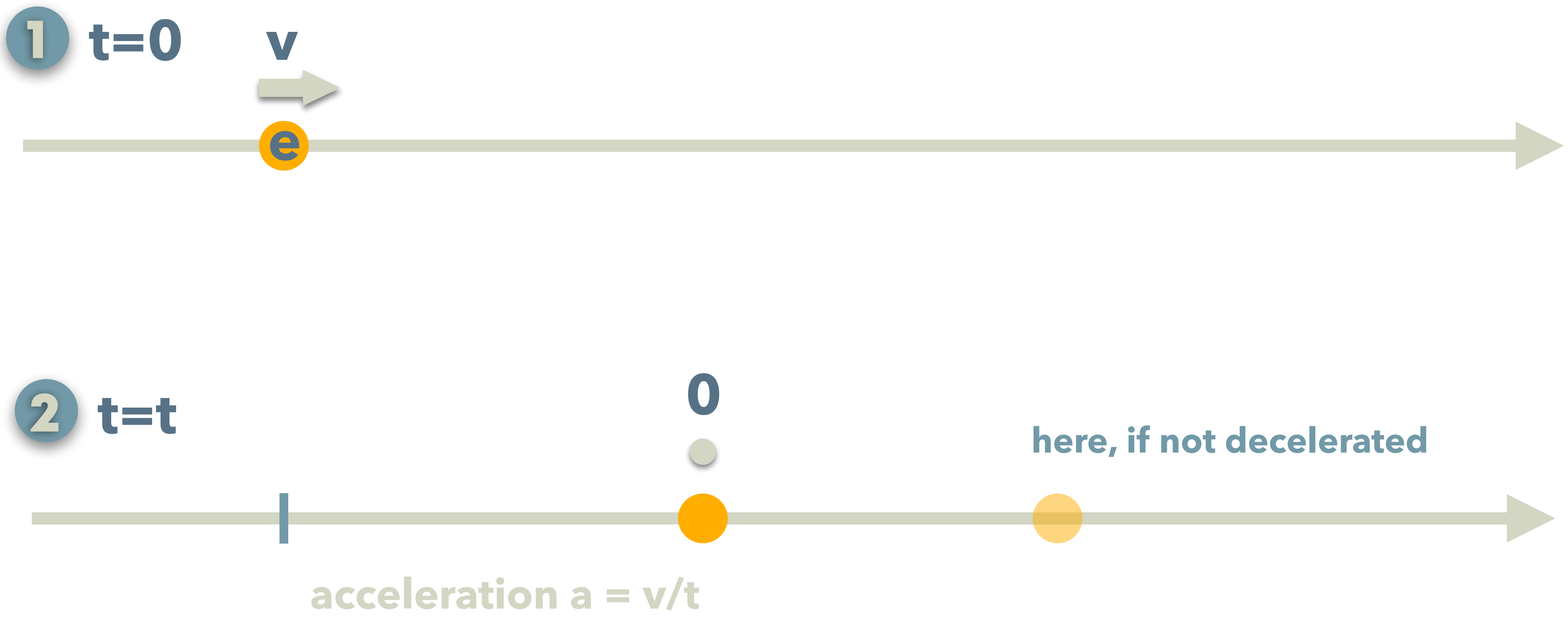


all B look similar
at Jeans side

low T blackbody
always lower flux
than high T

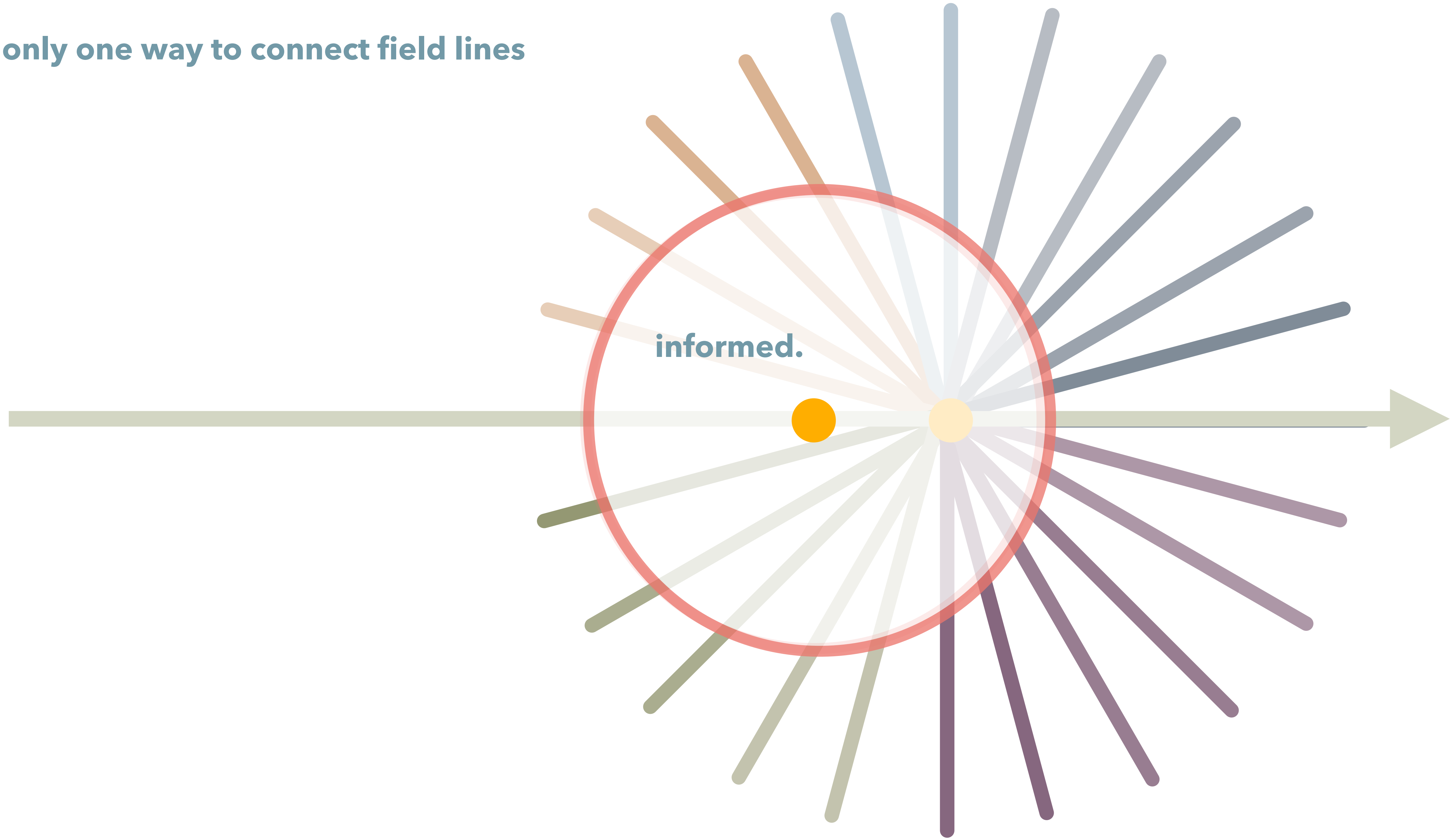
10 K	**300 μm**	sub-mm	gas, clouds dust
100 K	**30 μm**	mid-IR	molecular torus of AGN protoplanetary disk
1000	**3 μm**	near-IR	planet brown dwarfs
10000 K	**0.3 μm**	UV optical	reflected light of stars asteroids, planets stars (transit planet) stars (radial velocity)

Radiation



Radiative Processes in Astrophysics Rybicki & Lightman Section 3.2

Radiative Processes in Astrophysics Rybicki & Lightman Section 3.2



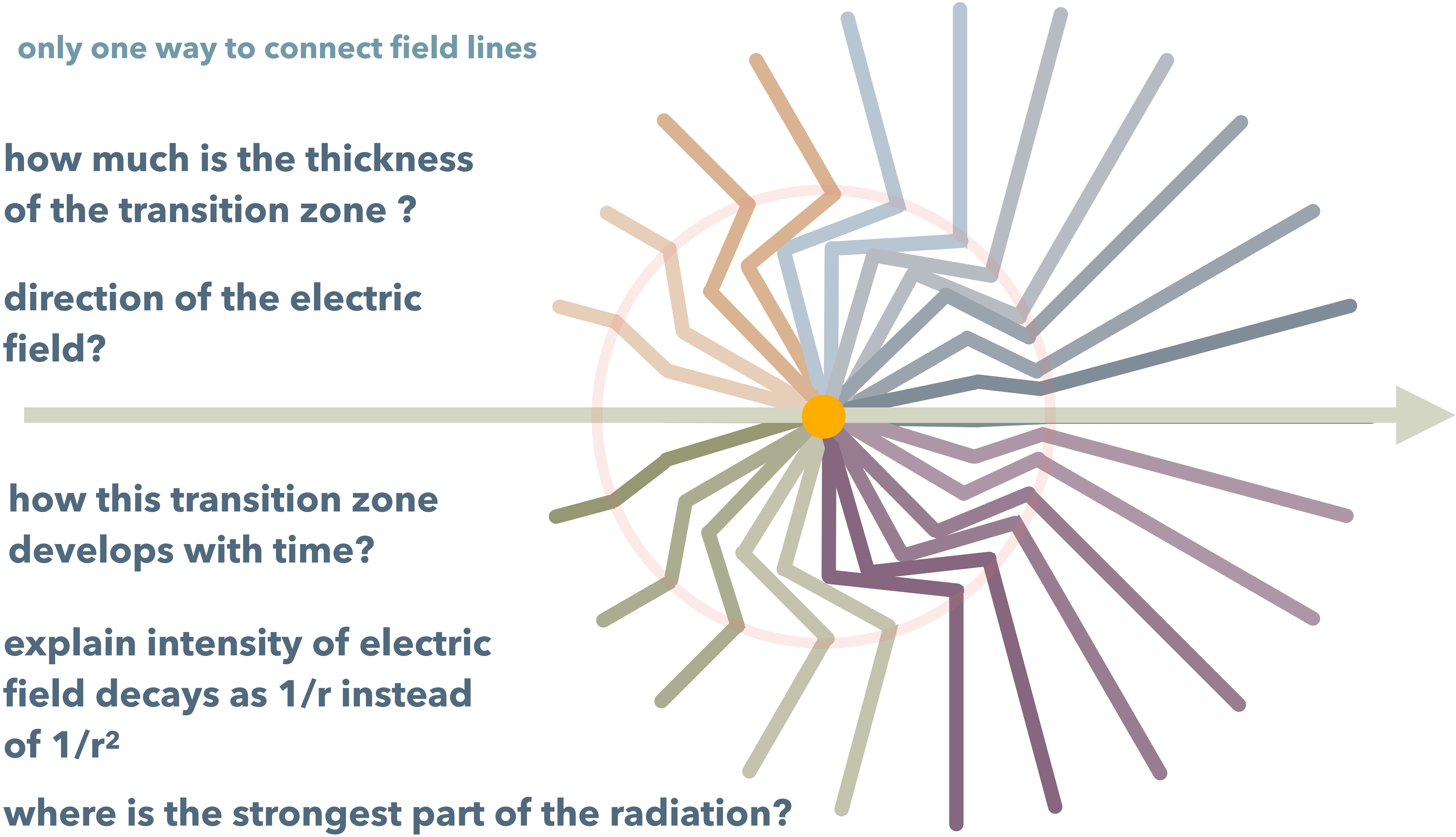
not informed that particle halted

only one way to connect field lines

informed.

Radiative Processes in Astrophysics Rybicki & Lightman Section 3.2

only one way to connect field lines



① how much is the thickness of the transition zone ?

② direction of the electric field?

③ how this transition zone develops with time?

④ explain intensity of electric field decays as 1/r instead of 1/r^2

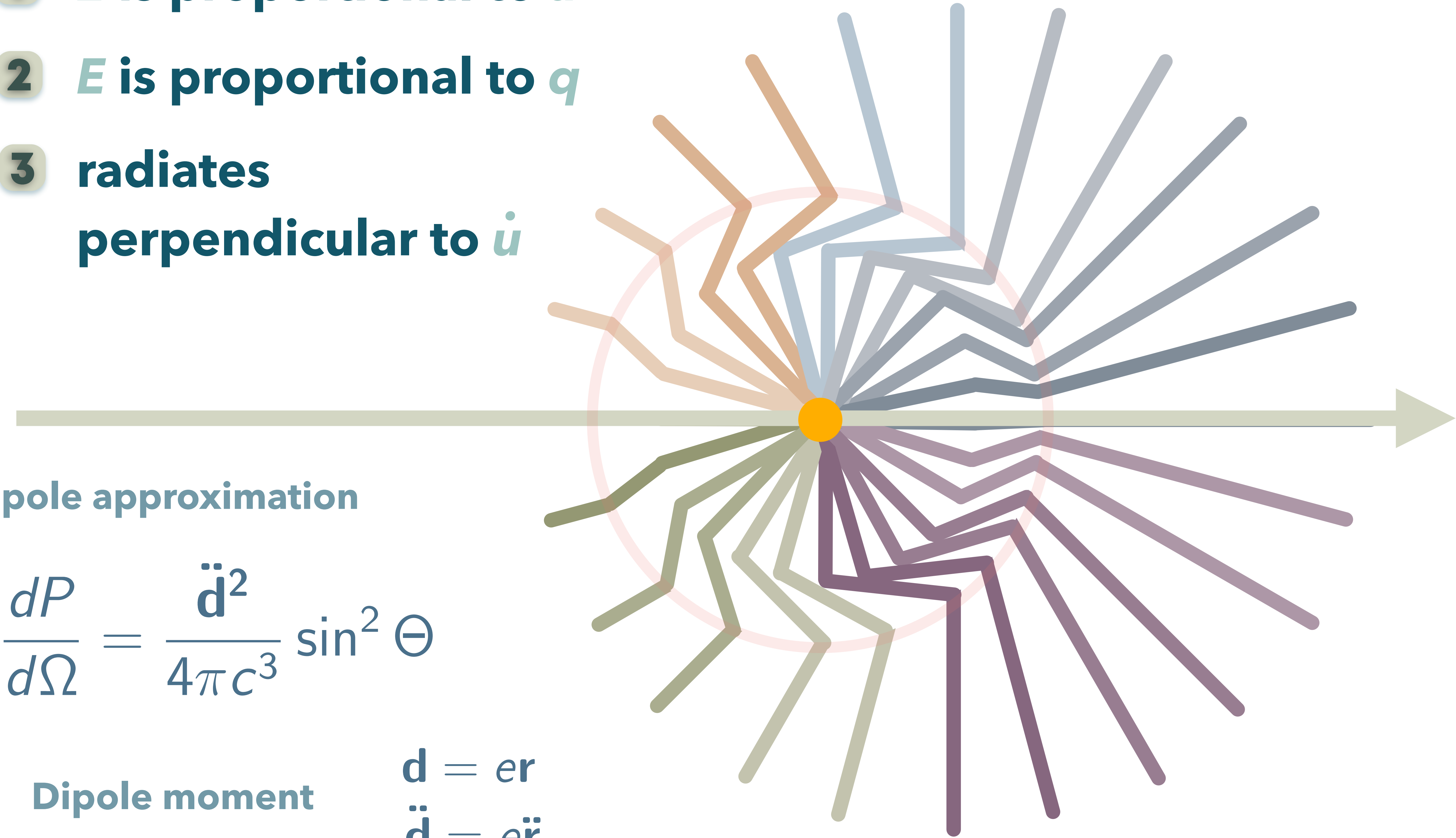
⑤ where is the strongest part of the radiation?

Radiative Processes in Astrophysics Rybicki & Lightman Section 3.2

Radiation

What we can take from the cartoon

1. E is proportional to $\dot{u}$
2. E is proportional to q
3. radiates perpendicular to $\dot{u}$



Dipole approximation

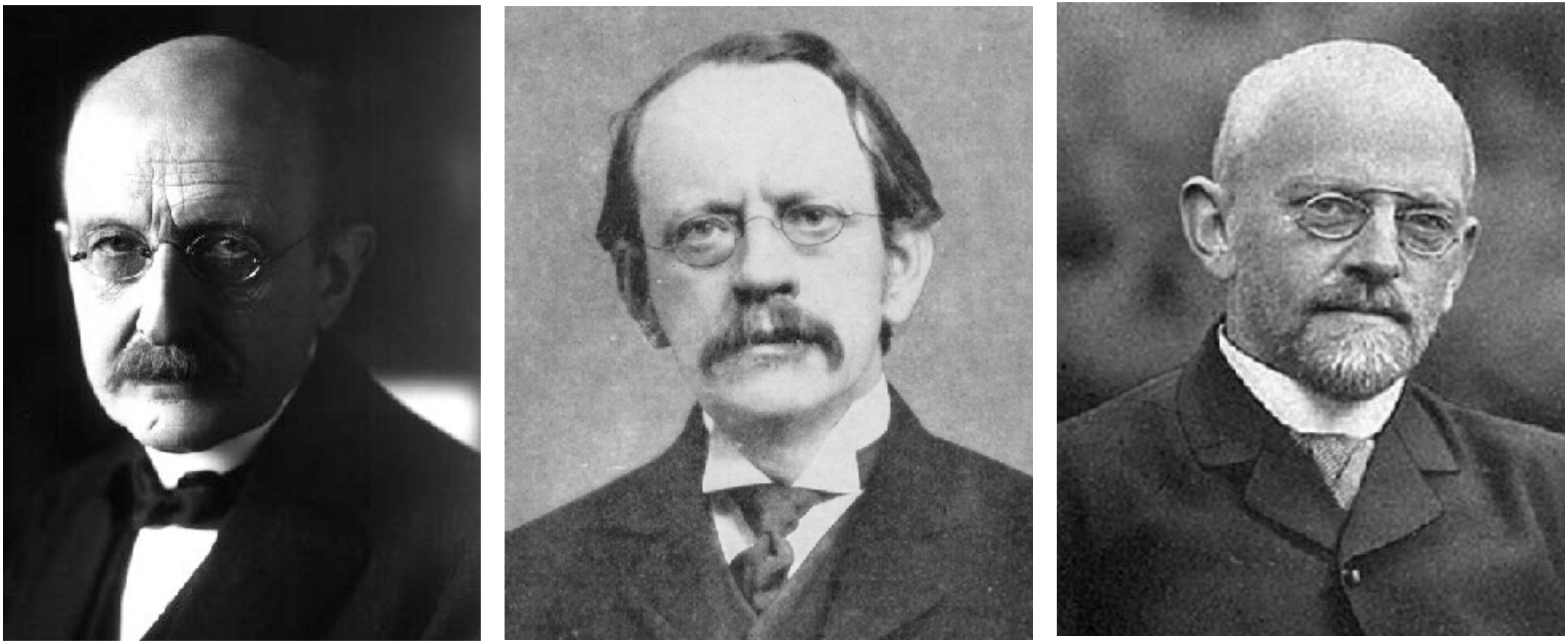
$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2 \Theta$$

Dipole moment

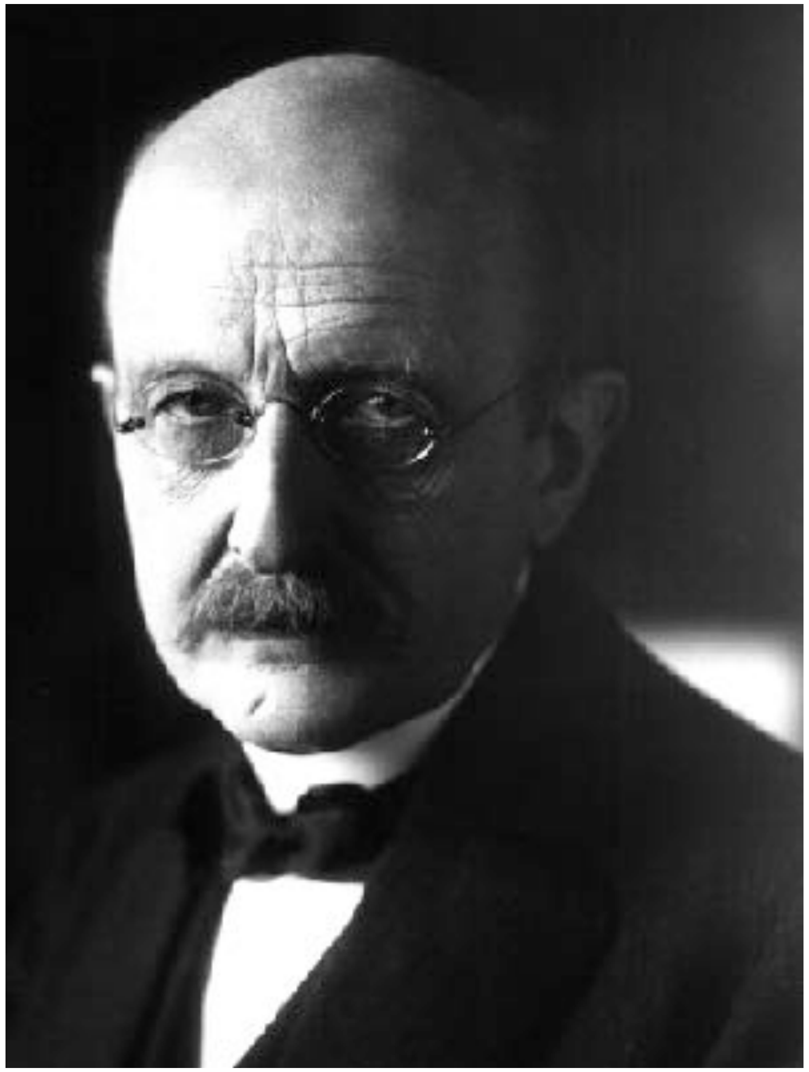
$$\mathbf{d} = e\mathbf{r}$$
$$\ddot{\mathbf{d}} = e\ddot{\mathbf{r}}$$

Radiative Processes in Astrophysics Rybicki & Lightman Section 3.2

Thomson scattering



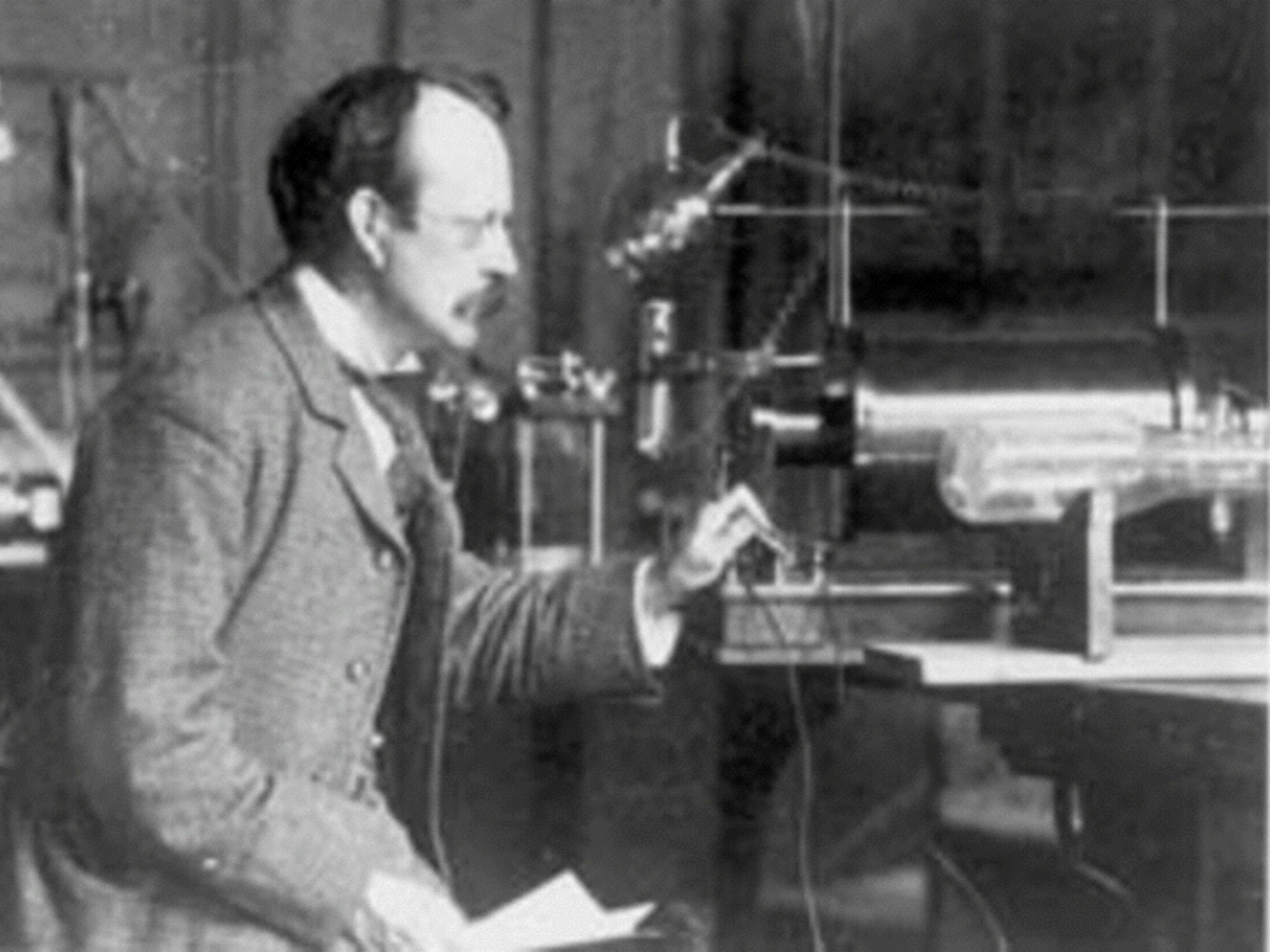
Thomson scattering

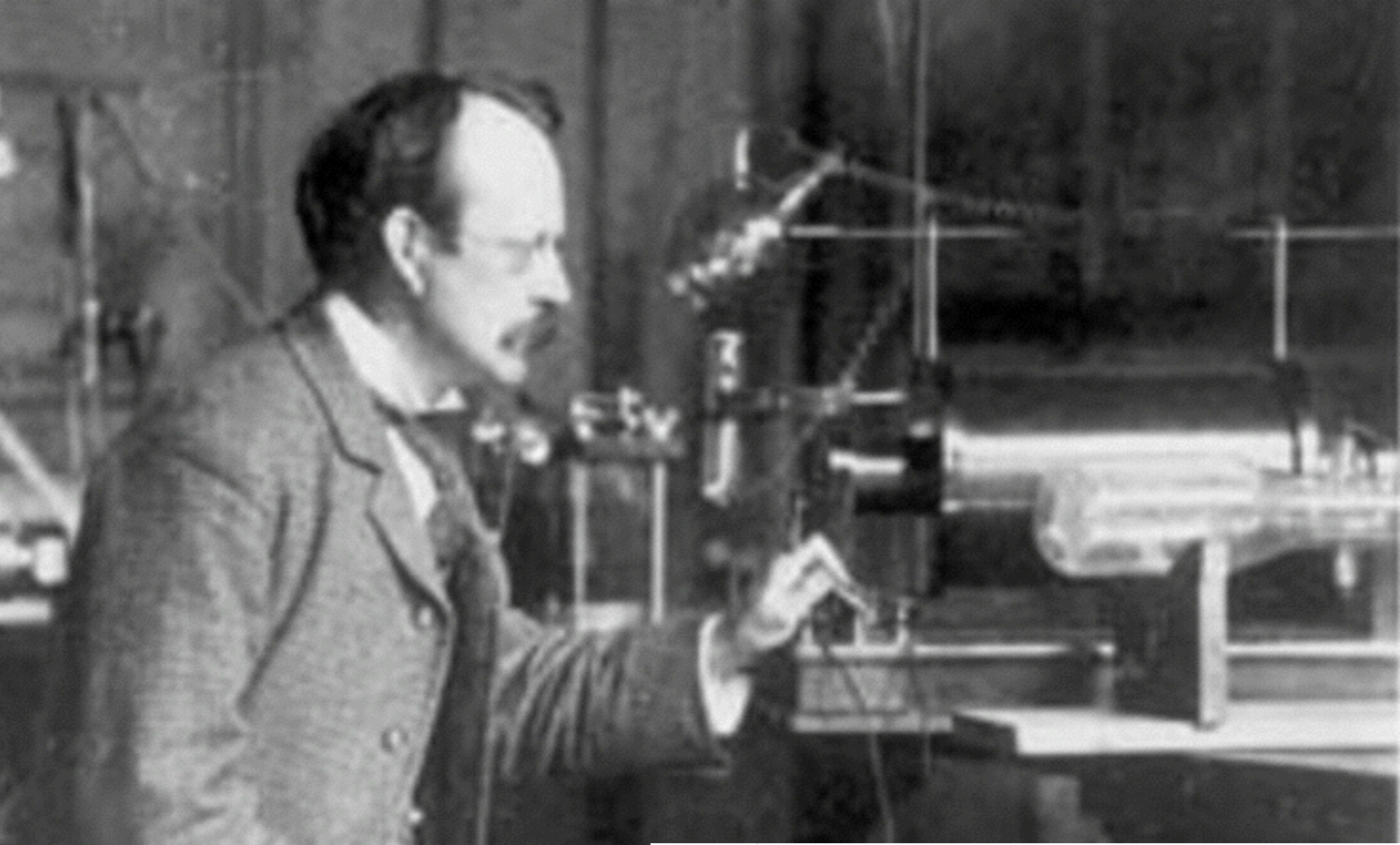


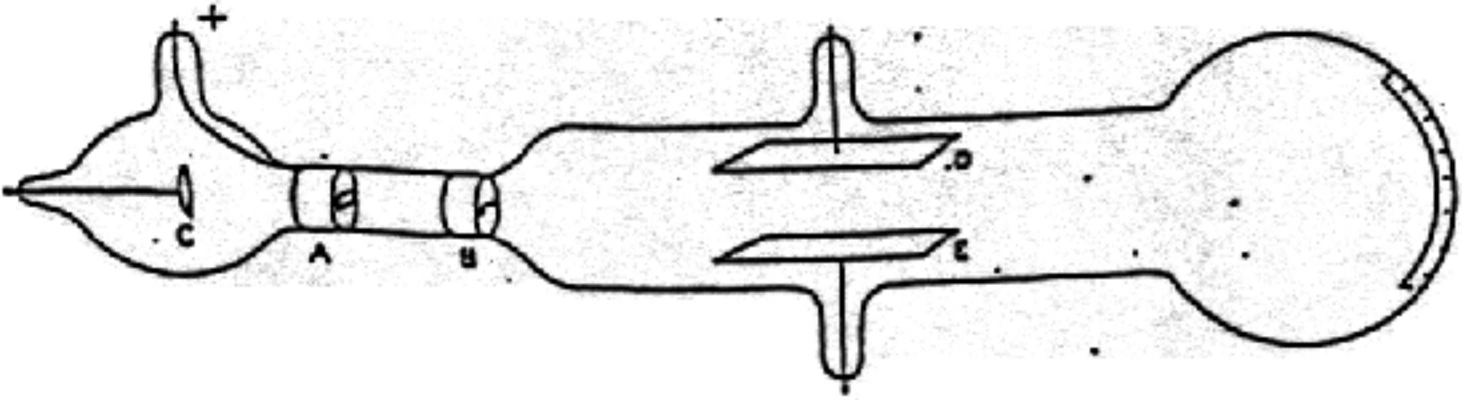
Max Planck

J. J. Thomson

David Hilbert



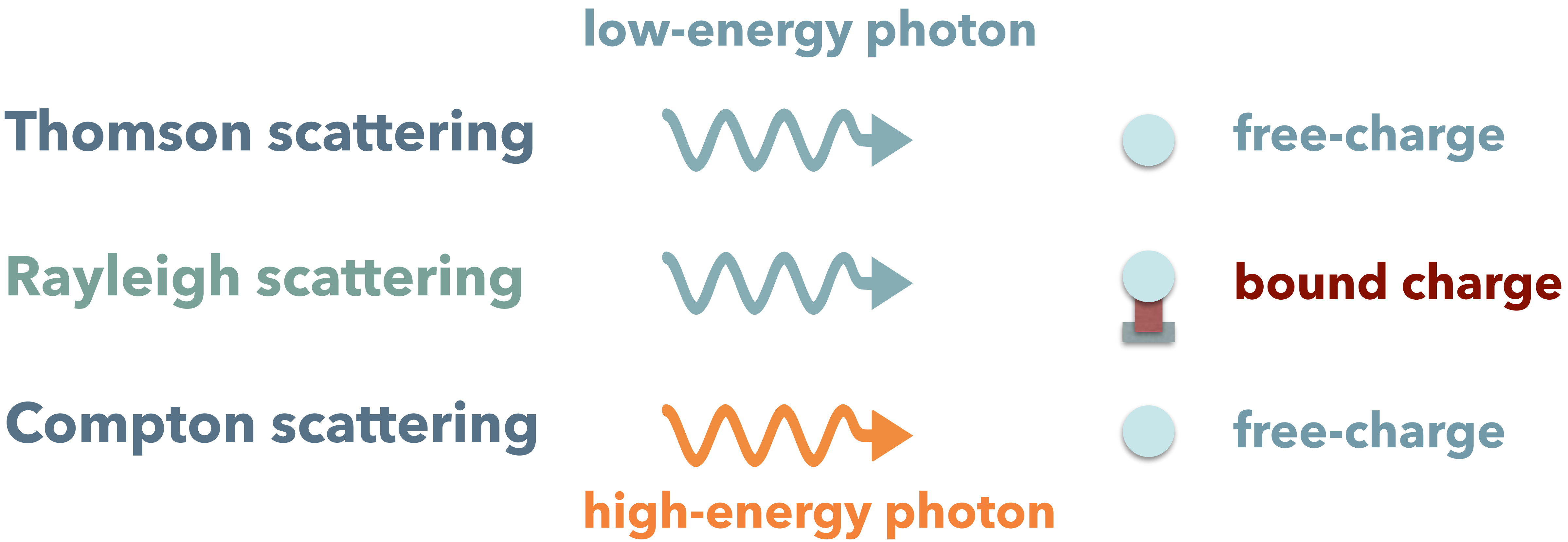



+
C
A
B
D
E

Where is the acceleration comes from?

something that pushes and pulls a charge?

Electromagnetic field!



low-energy photon

Thomson scattering free-charge

Rayleigh scattering **bound charge**

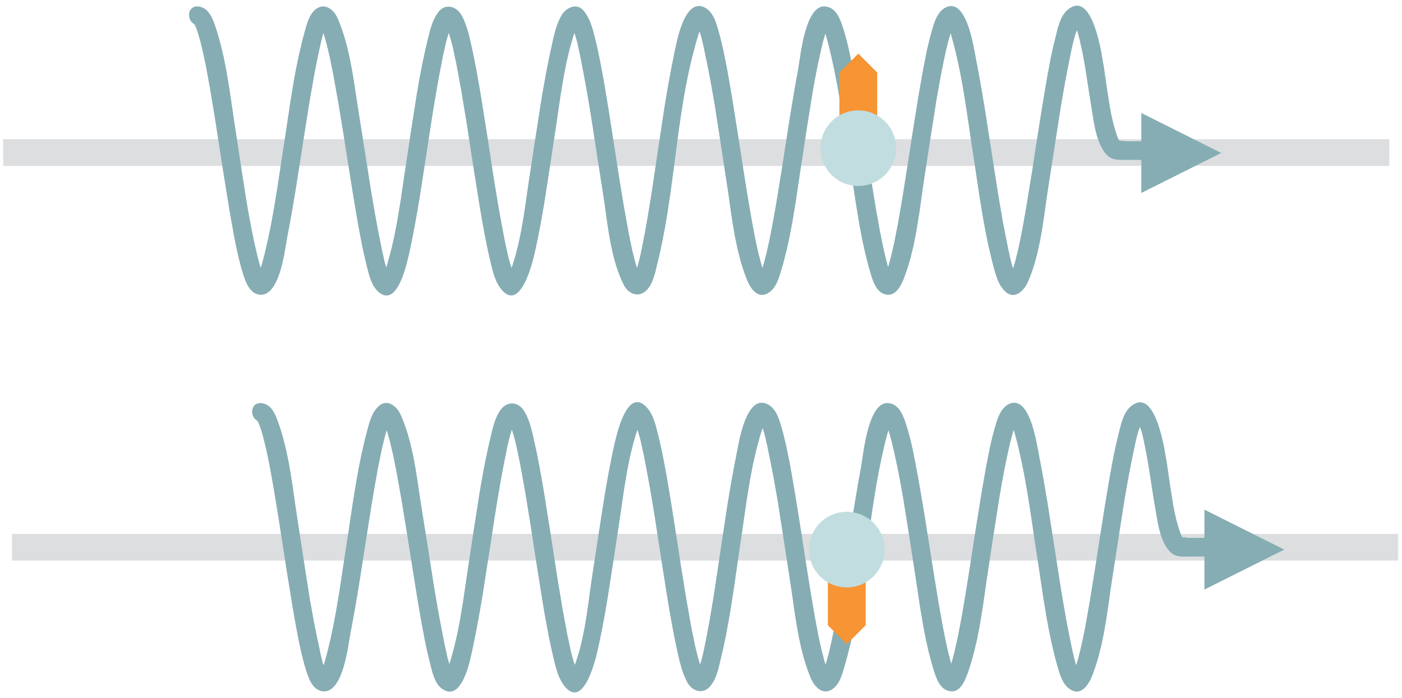
Compton scattering free-charge

high-energy photon

Rayleigh scattering

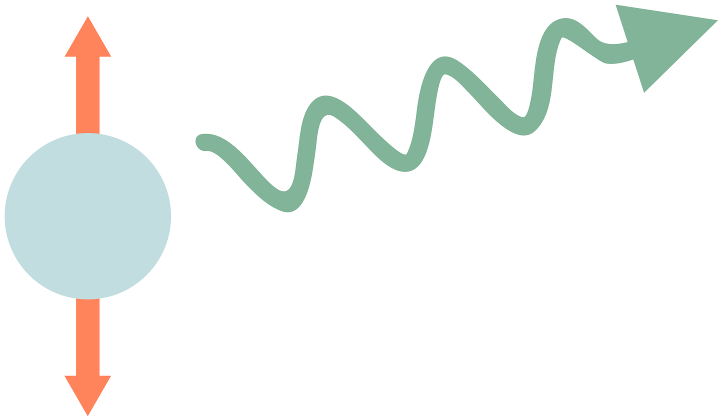
Dipole approximation

1. E is proportional to $\dot{u}$
2. E is proportional to q
3. radiates perpendicular to $\dot{u}$



Power

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta$$



Dipole approximation

$$\langle \sin^2 \omega t \rangle = \frac{1}{2}$$

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2 \Theta$$

incoming electric field

$$\mathbf{E} = \boldsymbol{\varepsilon} E_0 \sin \omega t$$

$$m\ddot{\mathbf{r}} = e\mathbf{E}$$

dipole moment

$$\mathbf{d} = e\mathbf{r}$$

$$\ddot{\mathbf{d}} = e\ddot{\mathbf{r}}$$

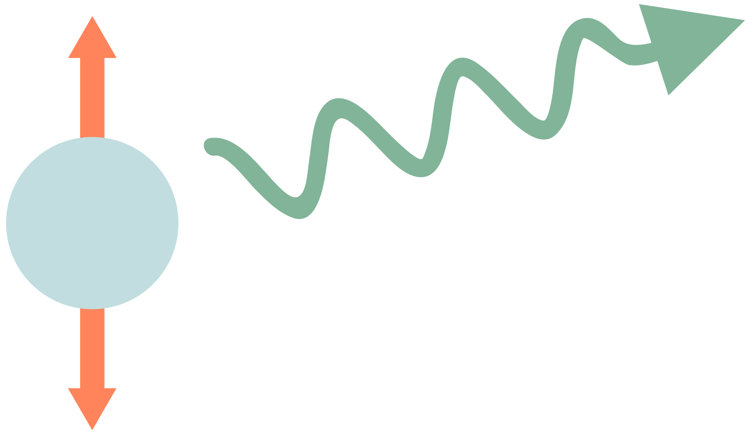
$$\ddot{\mathbf{d}} = \frac{e^2\mathbf{E}}{m}$$

$$|\ddot{\mathbf{d}}|^2 = \frac{e^4 E_0^2}{2m^2}$$

Dipole approximation

$$|\ddot{\mathbf{d}}|^2 = \frac{e^4 E_0^2}{2m^2}$$

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2 \Theta$$



total energy scattered?

$$P = \int \frac{dP}{d\Omega} d\Omega$$

$$\int \sin^2 \Theta d\Omega = \int_0^{2\pi} \int_0^{\pi} \sin^2 \Theta \sin \Theta d\Theta d\Phi$$

$$= 2\pi \int_0^{\pi} (1 - \cos^2 \Theta) \sin \Theta d\Theta$$

$$\frac{dP}{d\Omega} = \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2 \Theta$$

$$\int_0^{\pi} (1 - \cos^2 \Theta) \sin \Theta d\Theta = -\left[\cos \Theta\right]_0^{\pi} + \left[\frac{\cos^3 \Theta}{3}\right]_0^{\pi}$$

$$= \left[-(-1-1) + \frac{-1}{3} - \frac{1}{3}\right]$$

$$= \frac{4}{3}$$

$$P = \frac{e^4 E_0^2}{8\pi m^2 c^3} \cdot 2\pi \frac{4}{3}$$

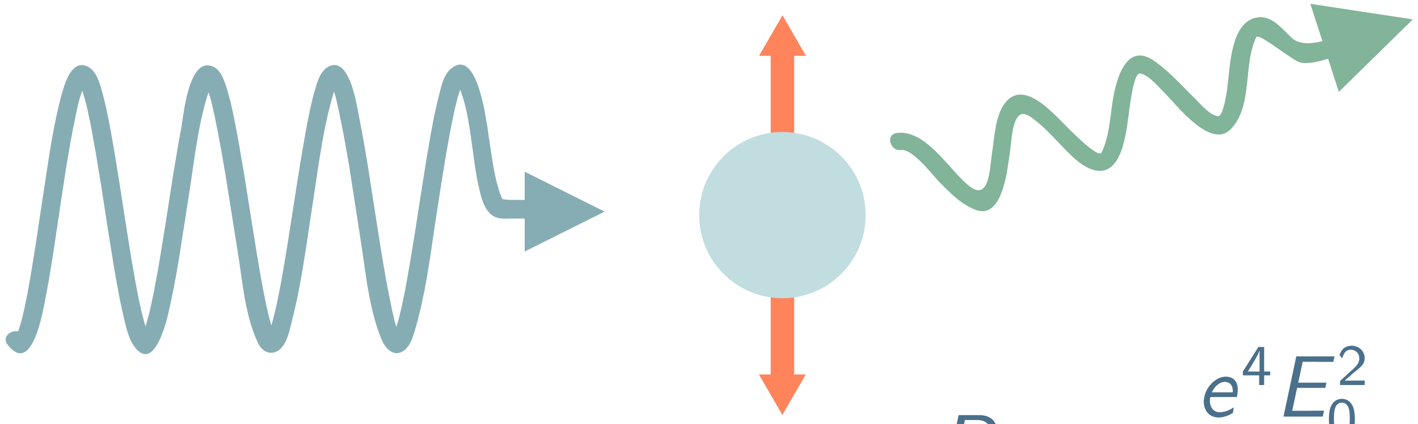
$$= \frac{e^4 E_0^2}{3 m^2 c^3}$$

Thomson cross section

$$P = \langle S \rangle \sigma_T$$

$$\sigma_T = \frac{P}{\langle S \rangle}$$

$$= \frac{8\pi}{3} \frac{e^4}{m^2 c^4}$$



$$\langle S \rangle = \frac{c}{8\pi} E_0^2$$

$$P = \frac{e^4 E_0^2}{3 m^2 c^3}$$

Poynting vector

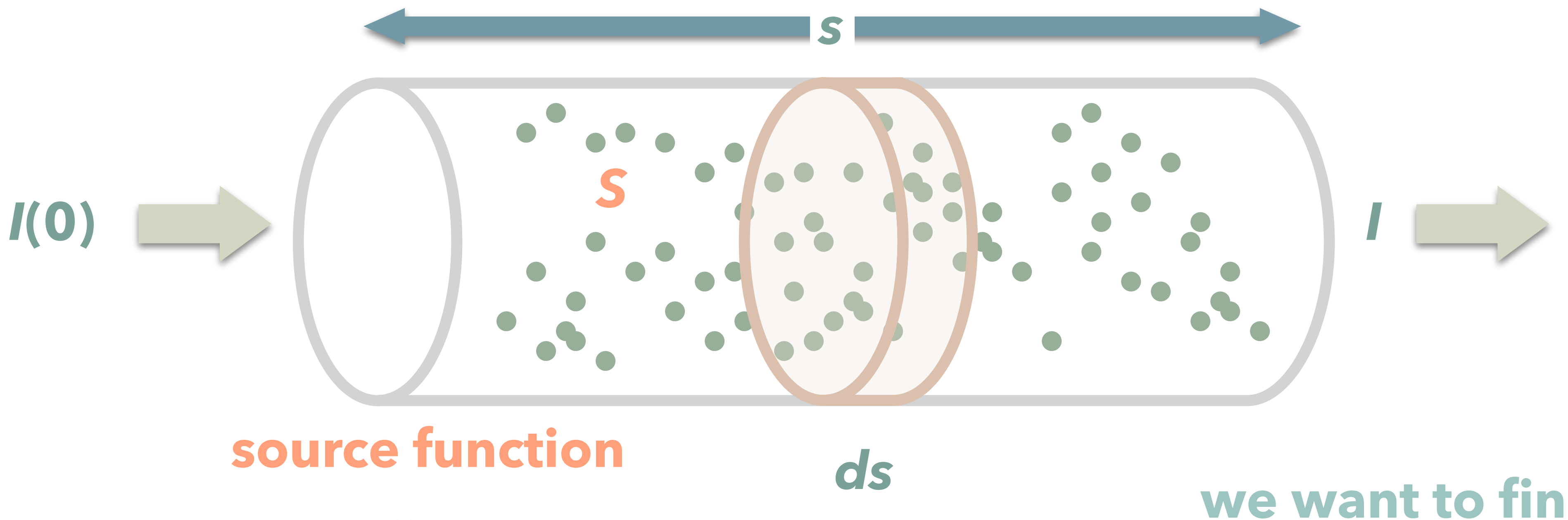
~specfic intensity I_ν

Thomson scattering

1. Scattering is a re-emission
2. Frequency independent
3. Polarized as in the incident light
4. Forward scattering

② Why Thomson scattering is sometims referred to "electron" scattering?

Radiation transfer



absorption coefficient

$$\frac{dI}{ds} = -aI + j$$

emission coefficient

$a ds = d\tau$ **optical depth**

$S = \frac{j}{a}$ source function

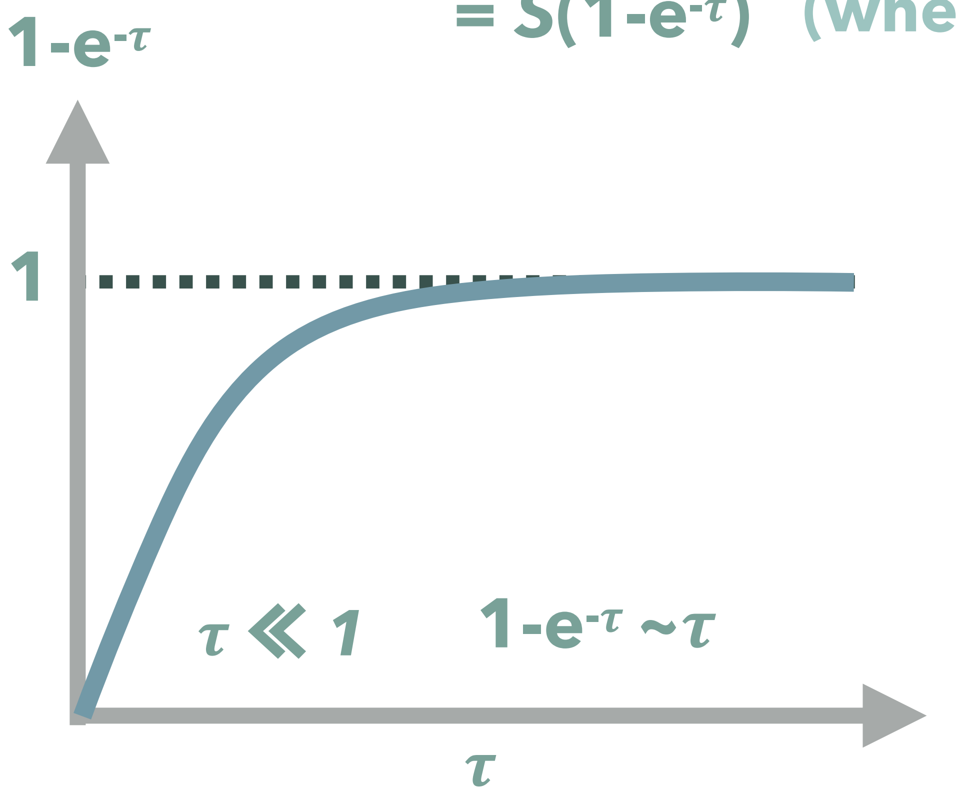
$$\frac{dI}{d\tau} = -I + S$$

$$I = I(0)\, e^{-\tau} + S(1-e^{-\tau})$$

we want to find exit *I*

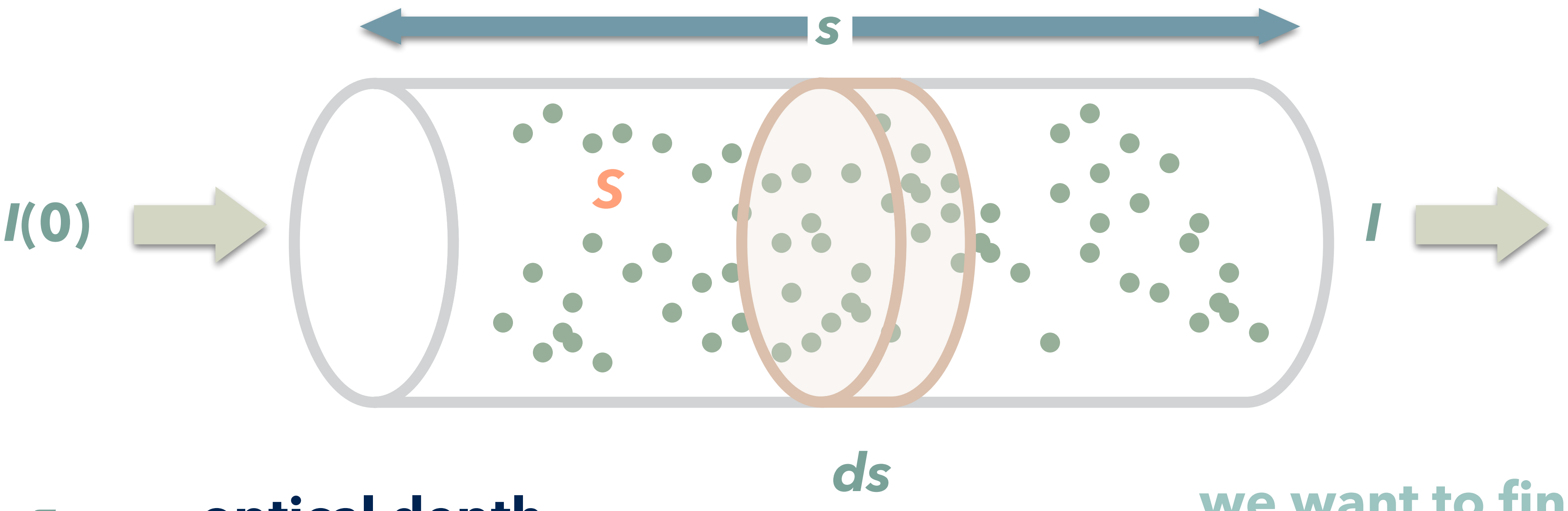
$I = \int j\, ds$ why not?

$= S(1-e^{-\tau})$ (when no bg)



③ Find above.

Radiation transfer



$as = \tau$ **optical depth**

$S = \frac{j}{a}$ source function

$$S(1-e^{-\tau}) = S\tau$$
$$= \frac{j}{a}\tau$$
$$= \frac{j}{a}as$$
$$= j\ s$$

optically thin case

indeed $I = j \cdot ds$

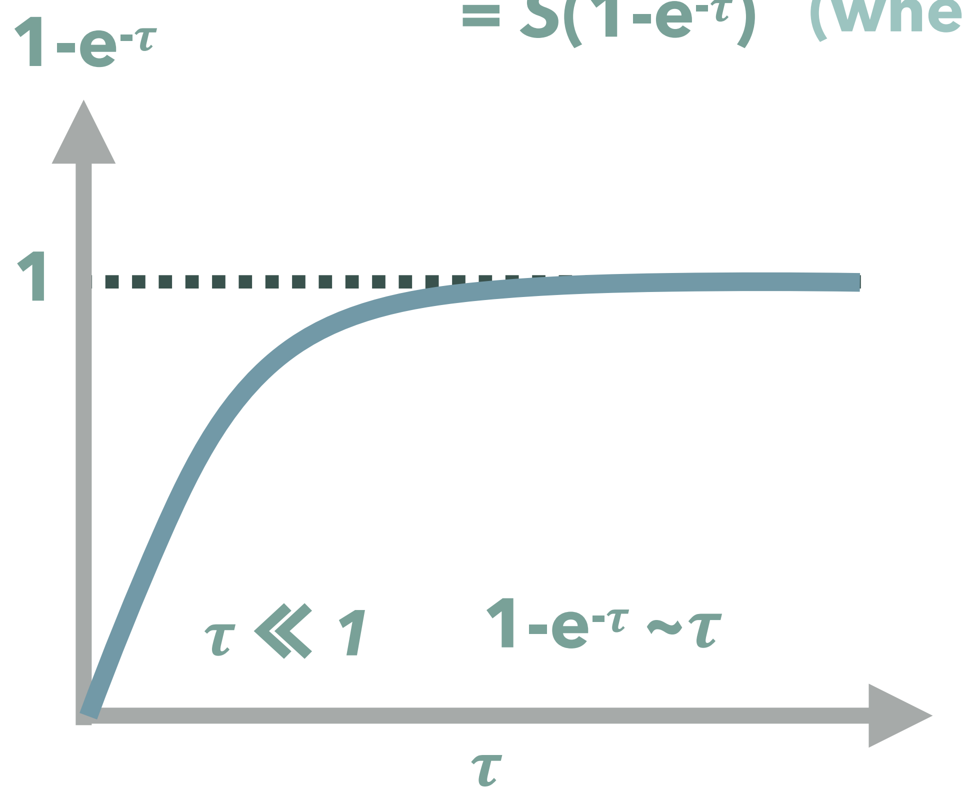
$a = n\sigma$

source function : j normalized by matter

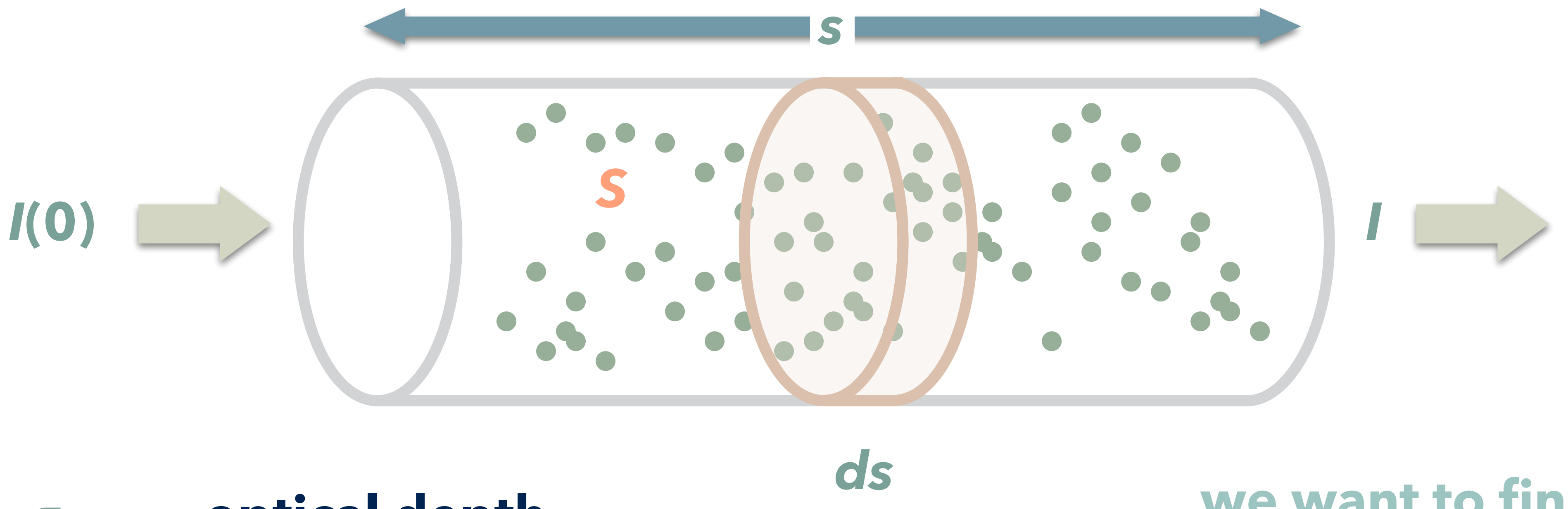
we want to find exit I

$I = \int j\,ds$ why not?

$= S(1-e^{-\tau})$ (when no bg)



Radiation transfer



$as = \tau$ **optical depth**

$S = \dfrac{j}{a}$ source function

$= B$ blackbody

Kirchhoff's law

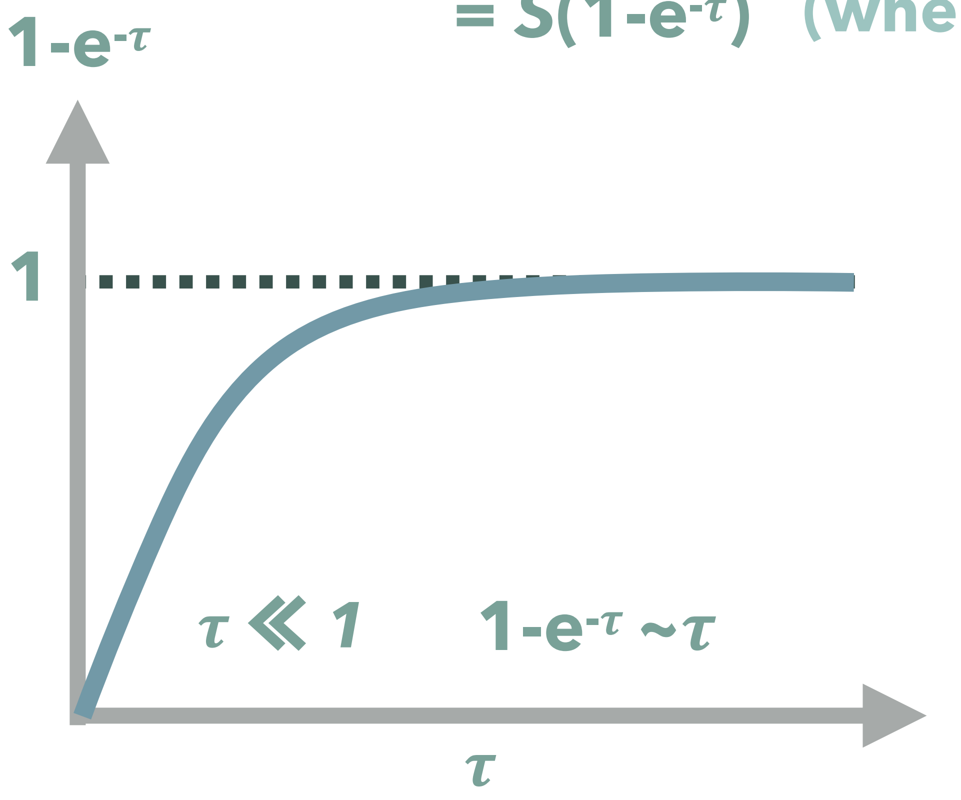
$I = I(0)\, e^{-\tau} + S(1-e^{-\tau})$

exit intensity
never overcomes blackbody

we want to find exit *I*

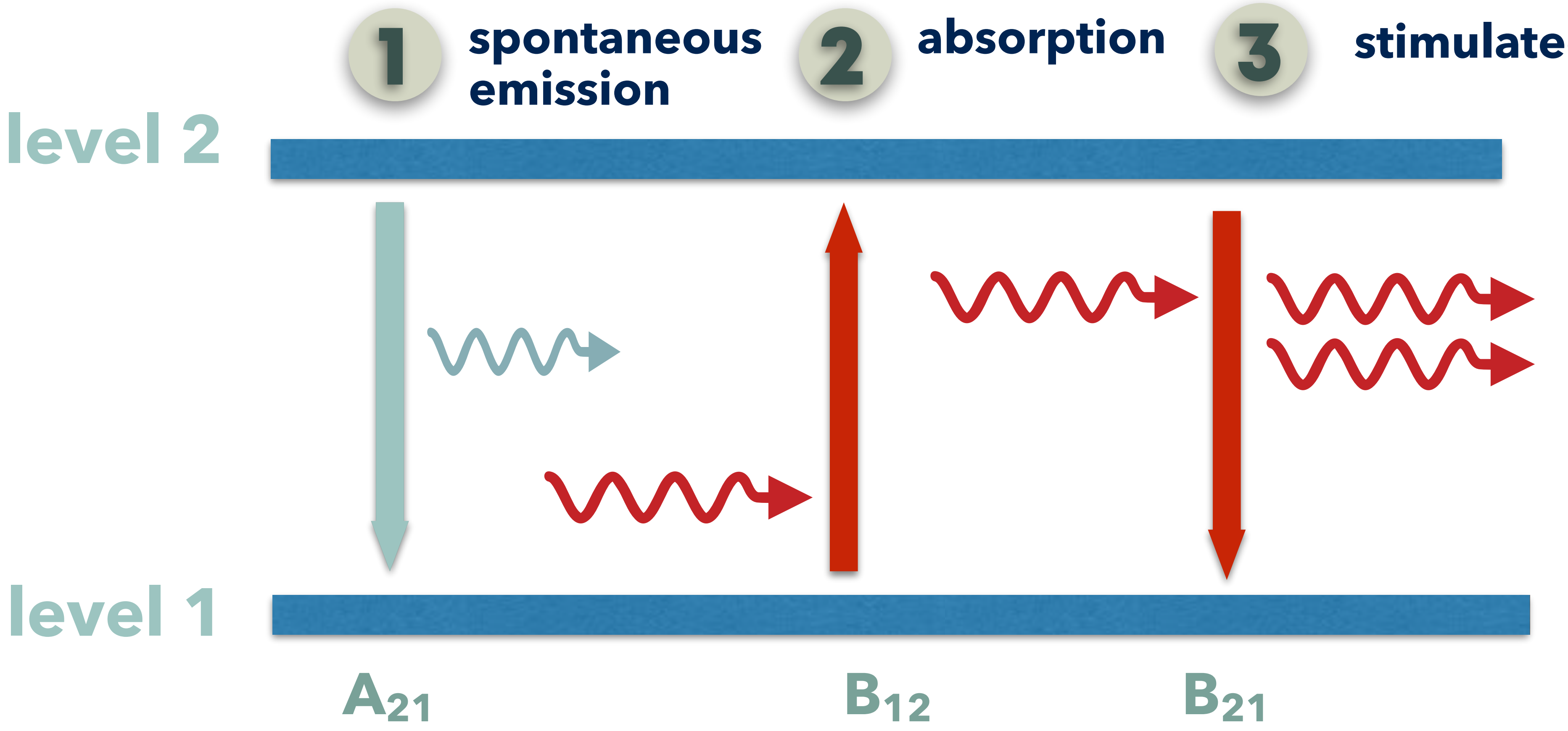
$I = \int j\, ds$ why not?

$= S(1-e^{-\tau})$ (when no bg)

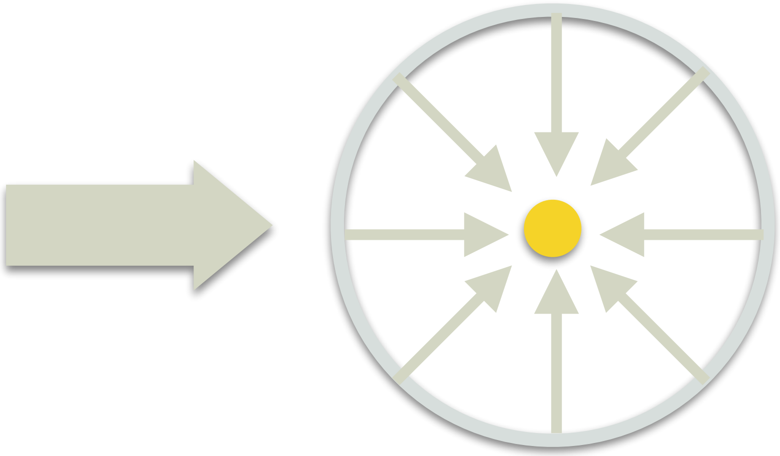


Einstein coefficient

how a molecule interact with radiation?



$$J = \frac{1}{4\pi}\int I \quad d\Omega$$



averaged out

system we are talking about

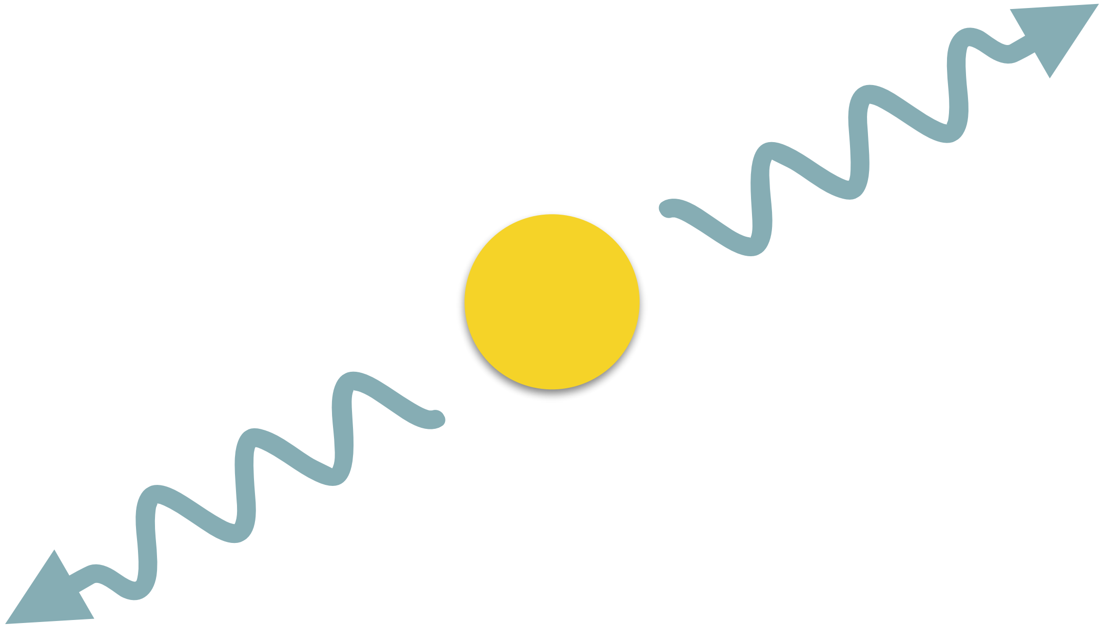
- no collision
- radiation only
- two levels
- thermal equilibrium

a molecule or an atom in level 2

A_{21} $[s^{-1}]$

spontaneous emission

you wait for one second.

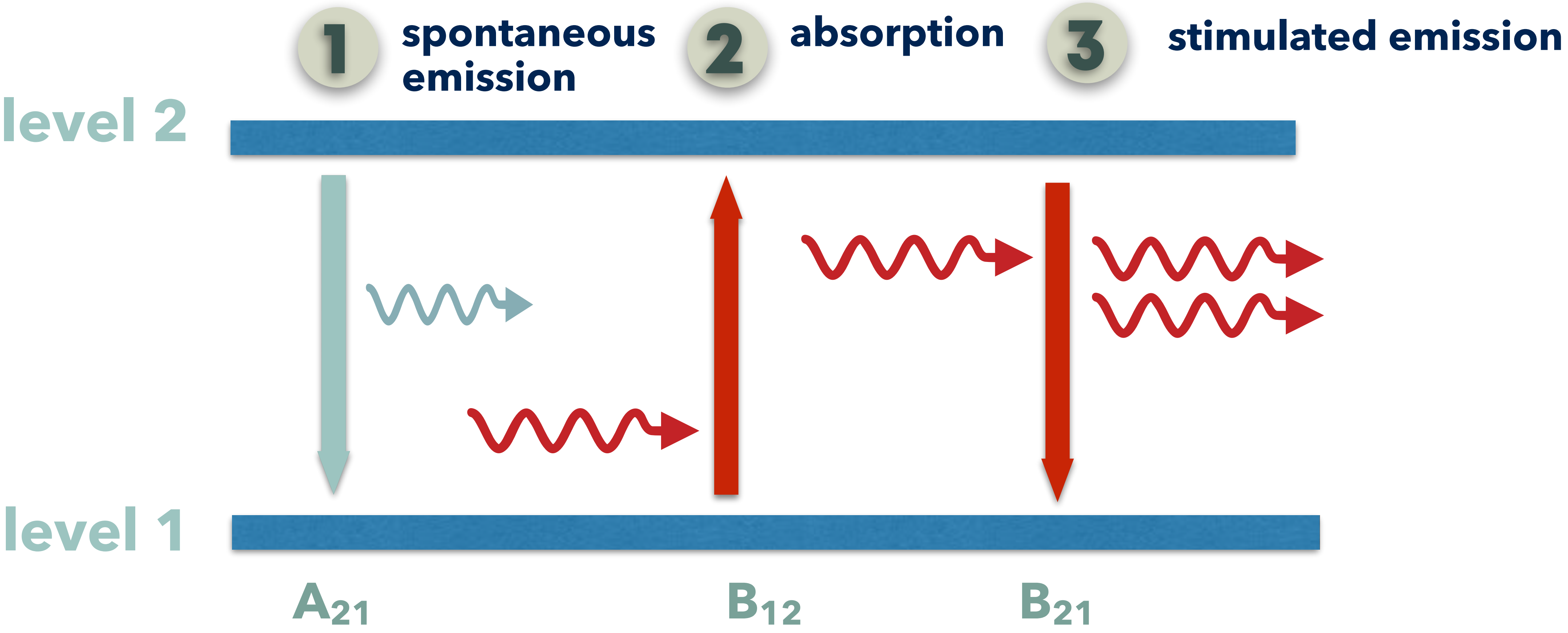


how often atoms / molecule decays to level 1

note it decays to 4 π

Einstein coefficient

how a molecule interact with radiation?



1 spontaneous emission

2 absorption

3 stimulated emission

level 2

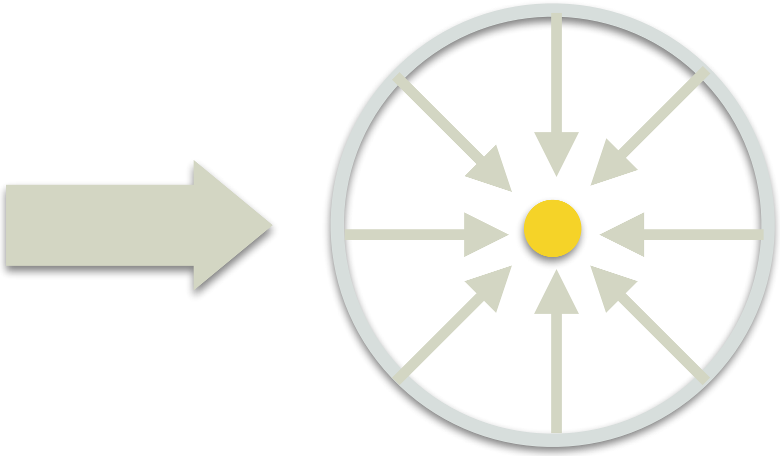
level 1

A_{21}

B_{12}

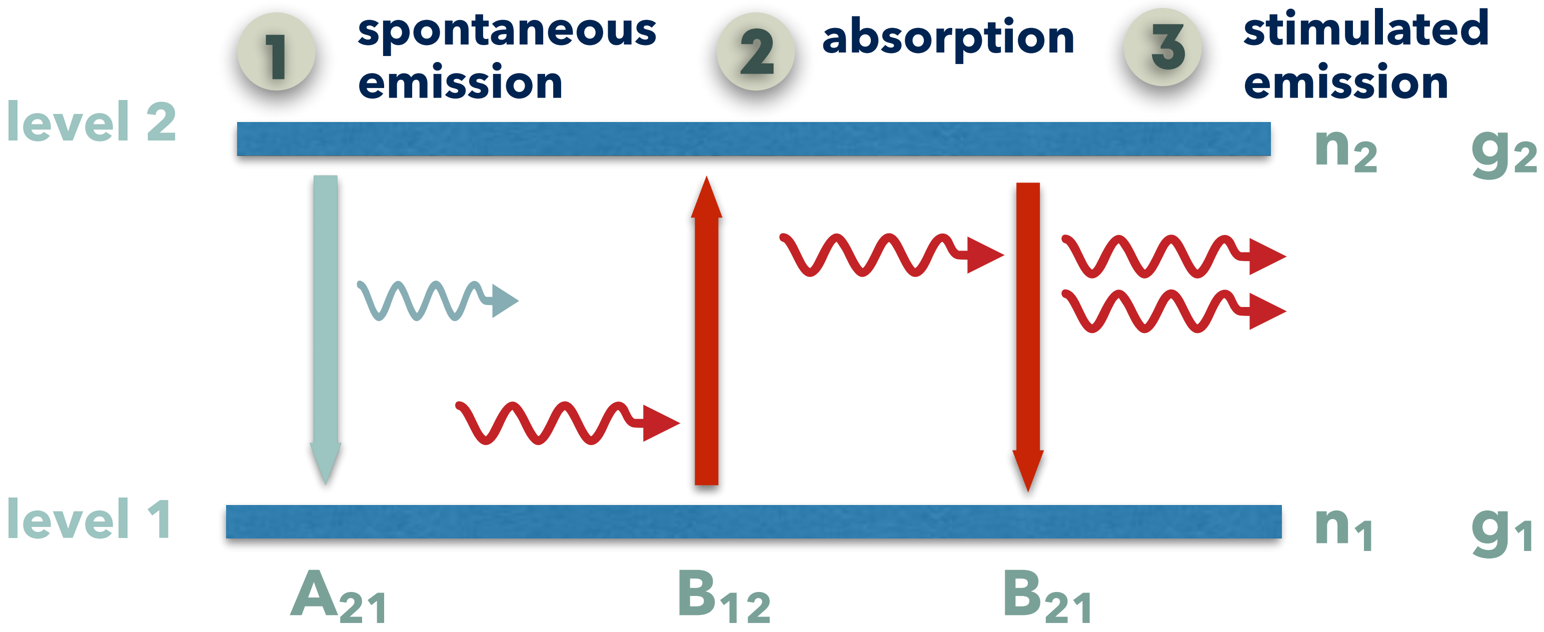
B_{21}

$$J = \frac{1}{4\pi}\int I \quad d\Omega$$



averaged out

$$\overbrace{Jn_1B_{12}}^{\text{up}} = \overbrace{n_2A_{21} + Jn_2B_{21}}^{\text{down}}$$



$$Jn_1 B_{12} = Jn_2 B_{21} + n_2 A_{21}$$

$$J\left(n_1 B_{12} - n_2 B_{21}\right) = n_2 A_{21}$$

$$J = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

$$J = \frac{\dfrac{n_2 A_{21}}{n_2 B_{21}}}{\dfrac{n_1 B_{12}}{n_2 B_{21}} - 1}$$

$$J = \frac{\dfrac{n_2 A_{21}}{n_2 B_{21}}}{\dfrac{g_1}{g_2}\dfrac{B_{12}}{B_{21}} \exp\left(\dfrac{\epsilon}{\tau}\right) - 1}$$

$$B = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\dfrac{h\nu}{kT}\right) - 1}$$

$$\frac{g_1}{g_2}\frac{B_{12}}{B_{21}} = 1$$

$$\frac{A_{21}}{B_{21}} = \frac{2h\nu^3}{c^2}$$

thermodynamical equilibrium

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left(-\frac{\epsilon}{\tau}\right)$$

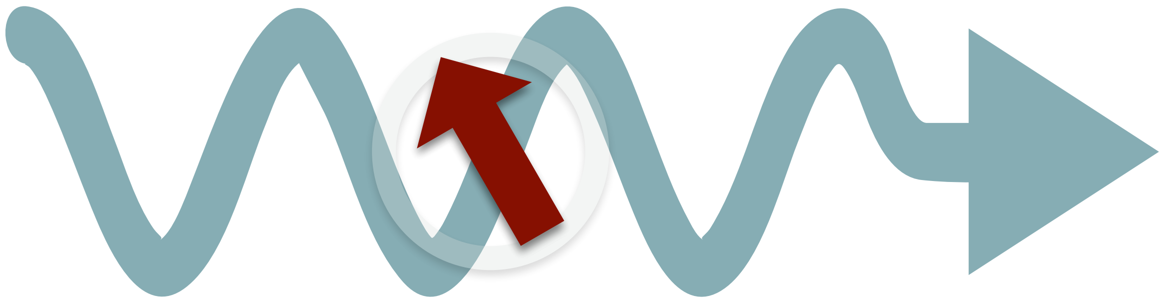
$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(\frac{\epsilon}{\tau}\right)$$

$$p \propto \exp\left(-\frac{\varepsilon_s}{\tau}\right)$$

$$A_{21} = \frac{64\pi^4\nu^3}{3hc^3}\mu^2\frac{J+1}{2J+3}$$

electric dipole moment

can be calculated from wavefunction



radiation can only interact with charge

(puting aside magnetic dipole moment)

Line formation

1. optical depth:

   when thick, we are always looking at the surface of τ ~1

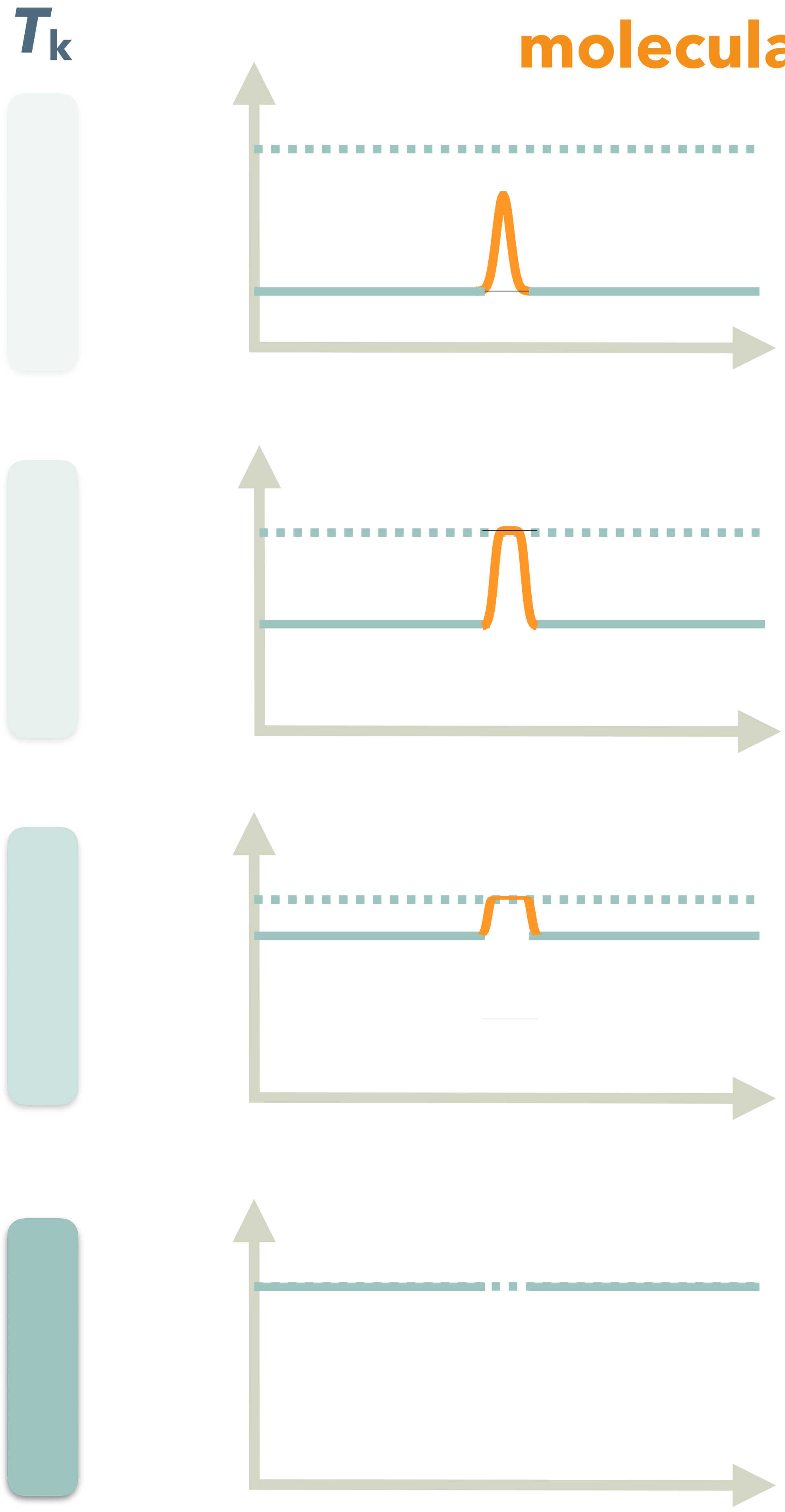
2. warmer blackbody has stronger radiation than cooler one over all frequency

3. when the line intensity reached to that blackgody at the frequency, cannot grow further.

Blackbody 11K > **CO J=1-0 line at 10 K**

molecular line $\tau \sim 0.1$

single temperature



T_k blackbody

gray body

$\tau \cdot B(T)$

$\tau \ll 1$

T_k

$\tau \sim 1$

$\tau \sim 0.1$

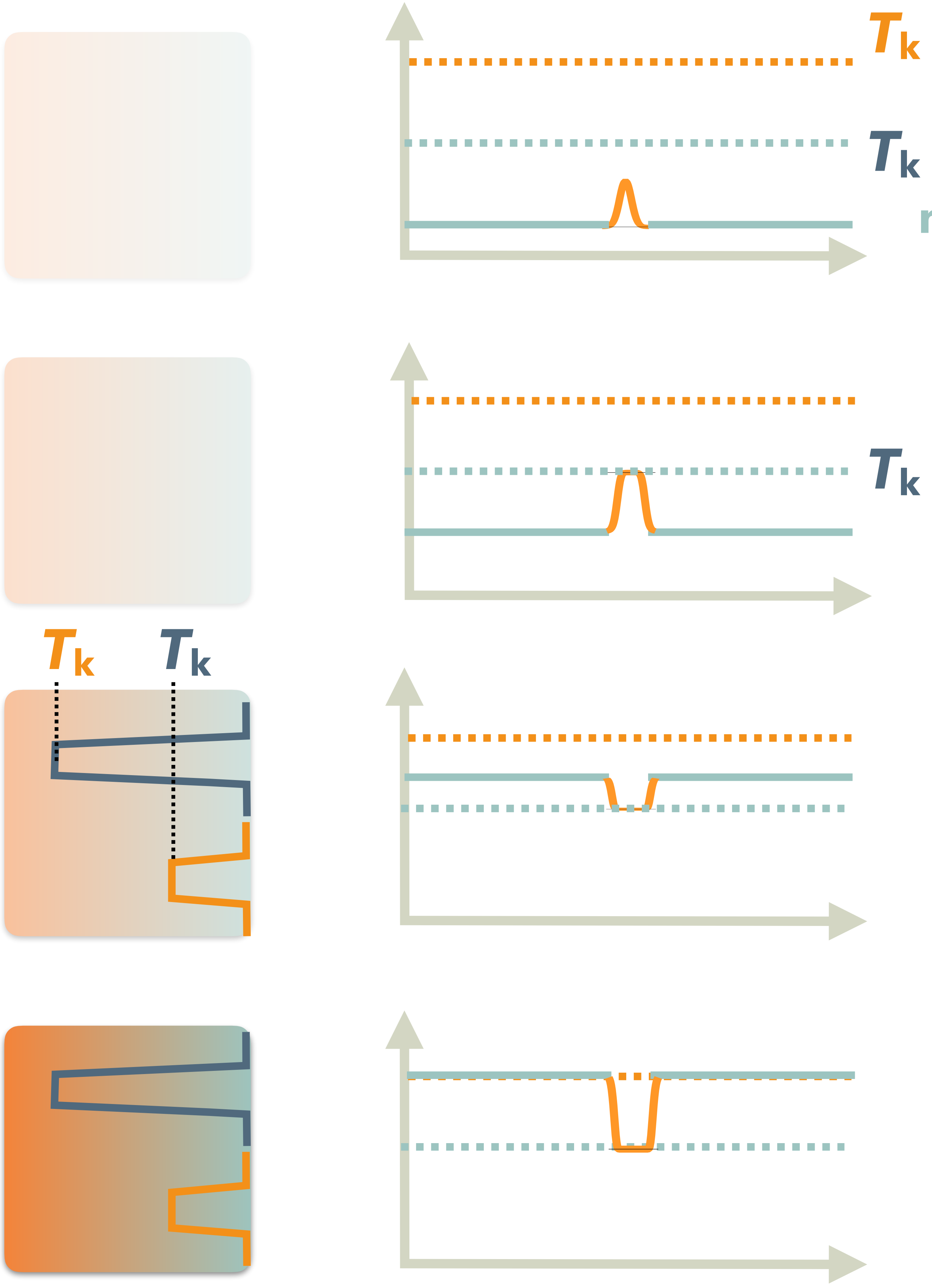
$\tau > 1$

$\tau \sim 1$

$\tau > 1$

$\tau > 1$

temperature gradient



$T_k > T_k$

blackbody

mixed gray body

$\tau \cdot B(T)$

$\tau \ll 1$

$\tau \sim 1$

$\tau \sim 0.1$

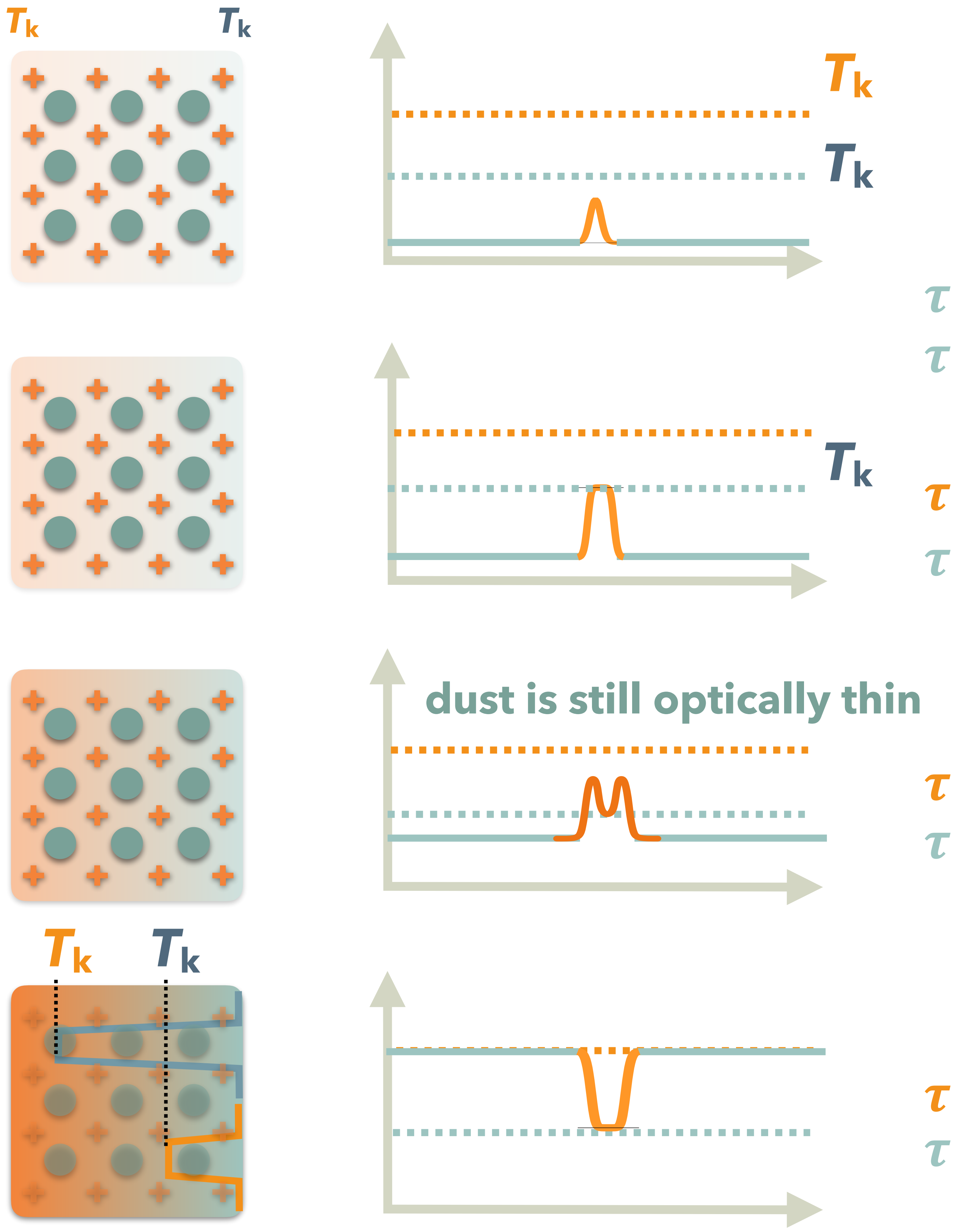
$\tau > 1$

$\tau \sim 1$

$\tau > \tau > 1$

● dust (blackbody)

✚ molecules (line)



$\tau \cdot B(T)$

$\tau \ll 1$

$\tau \sim 1$

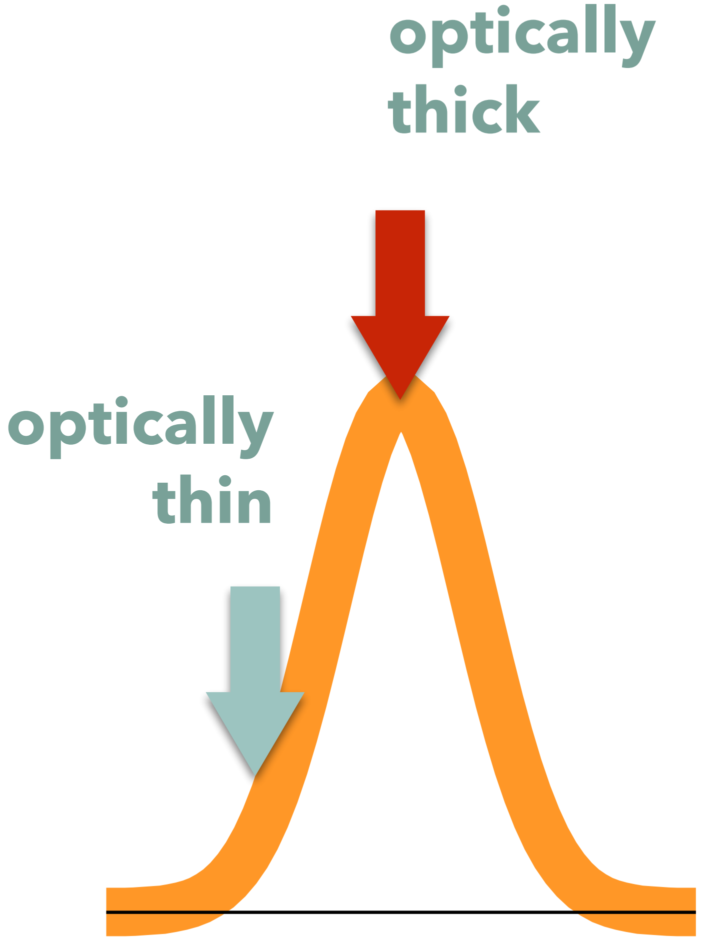
$\tau \sim 0.1$

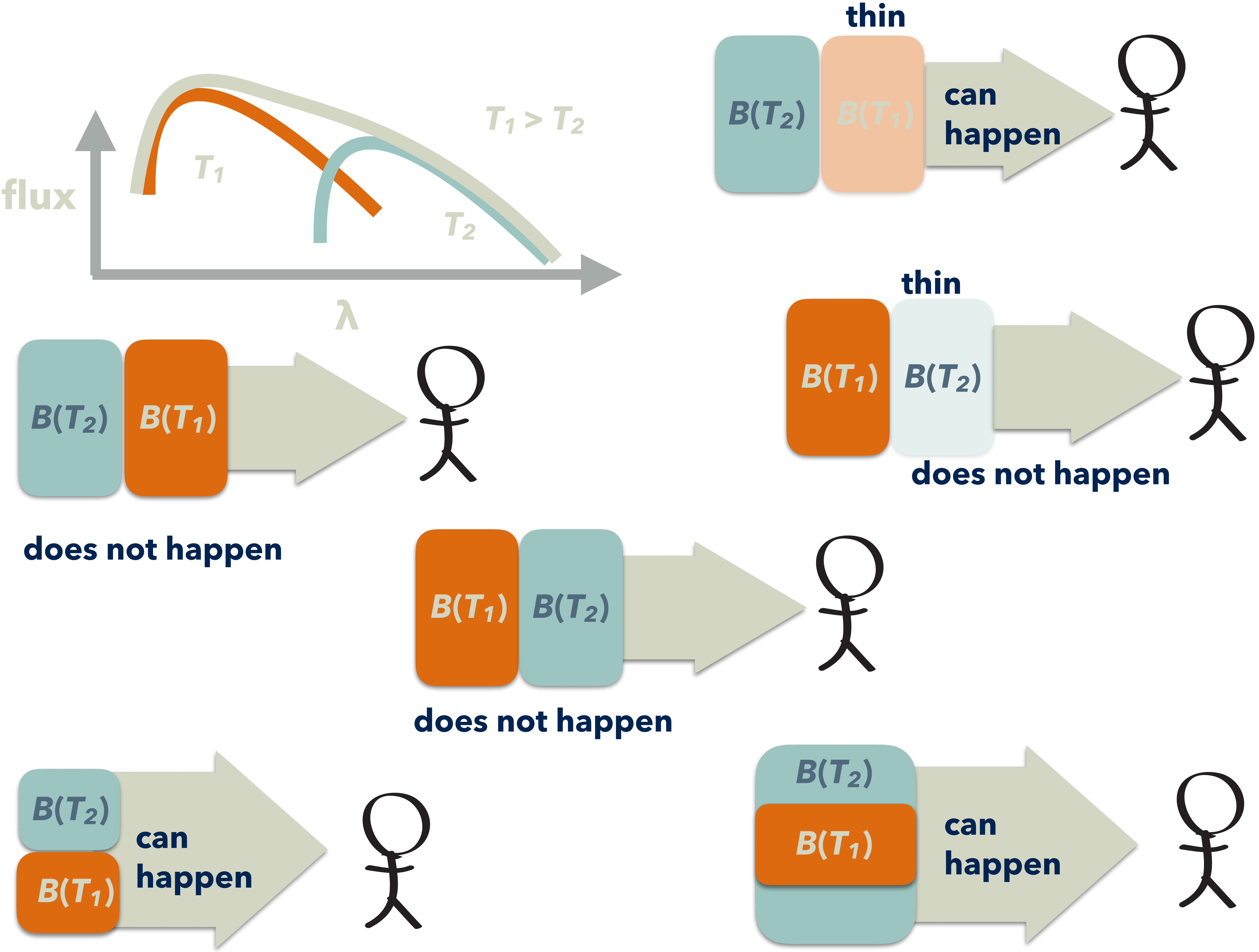
$\tau > 1$

$\tau \sim 1$

$\tau > 1$

$\tau > 1$





thin

$B(T_2)$ $B(T_1)$ can happen

thin

$B(T_1)$ $B(T_2)$

does not happen

$B(T_2)$ $B(T_1)$

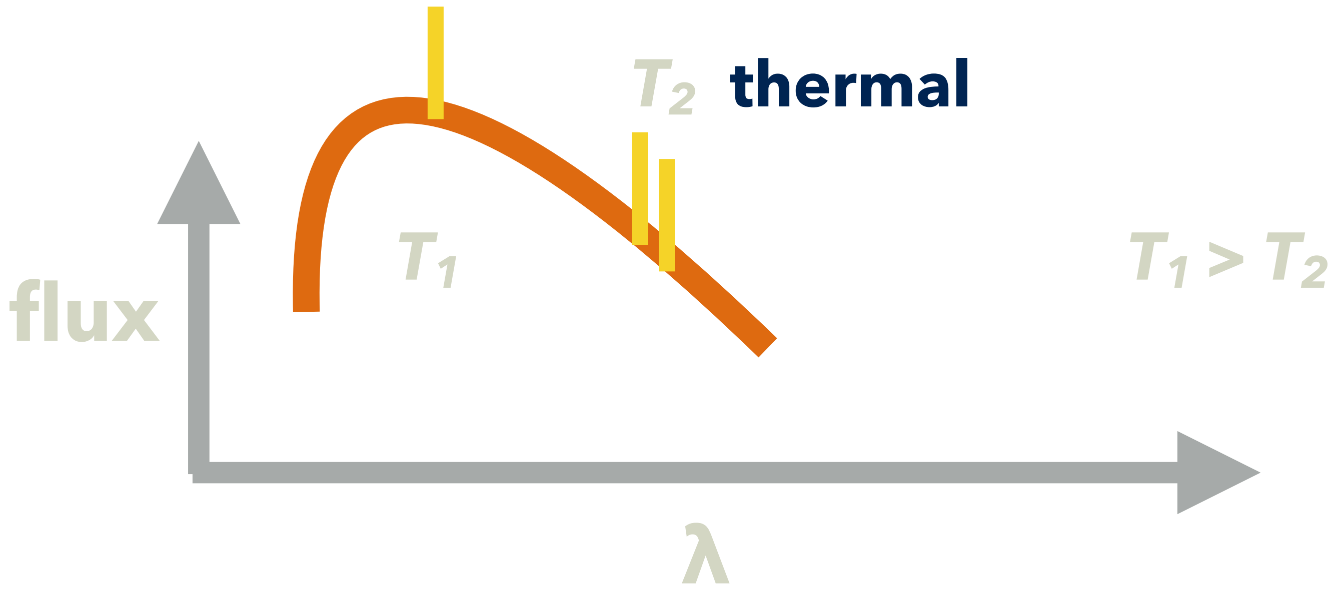
does not happen

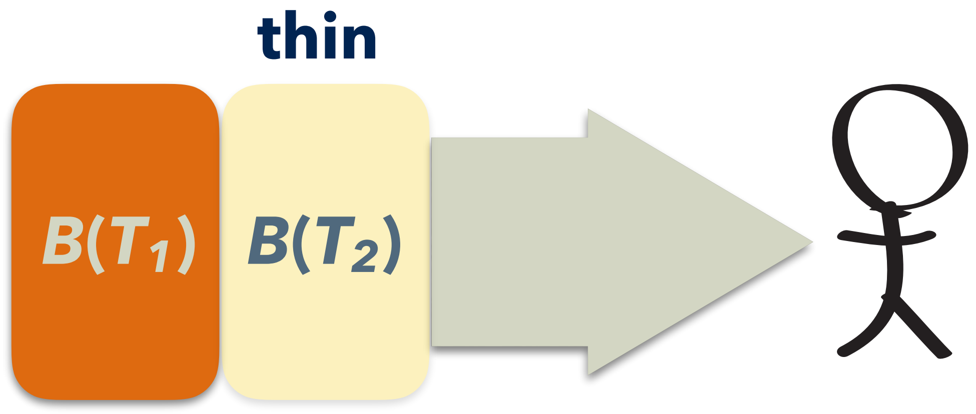
$B(T_1)$ $B(T_2)$

does not happen

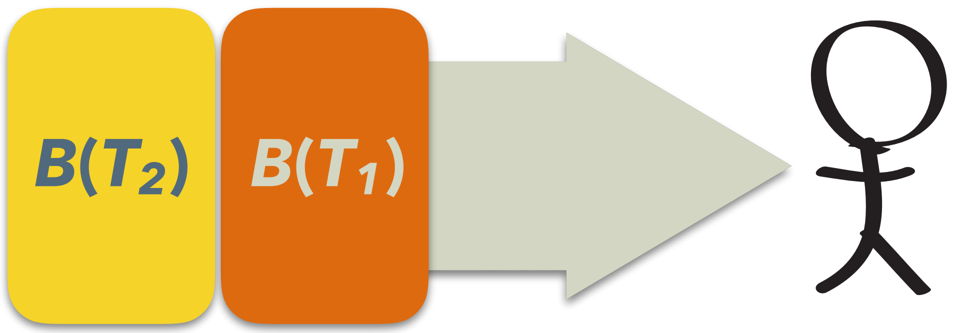
$B(T_2)$ $B(T_1)$ can happen

$B(T_2)$ $B(T_1)$ can happen

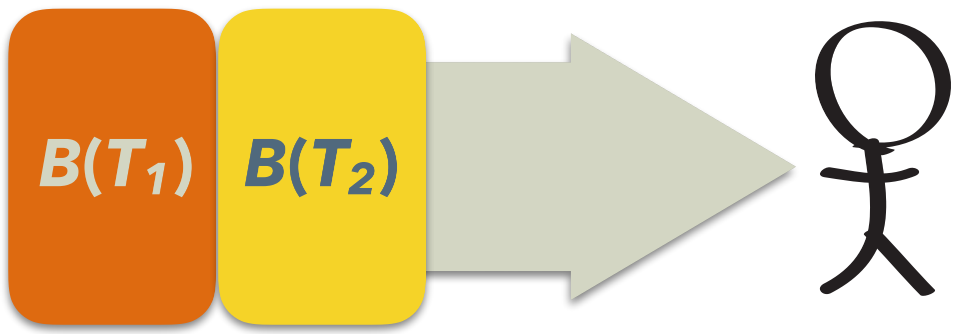




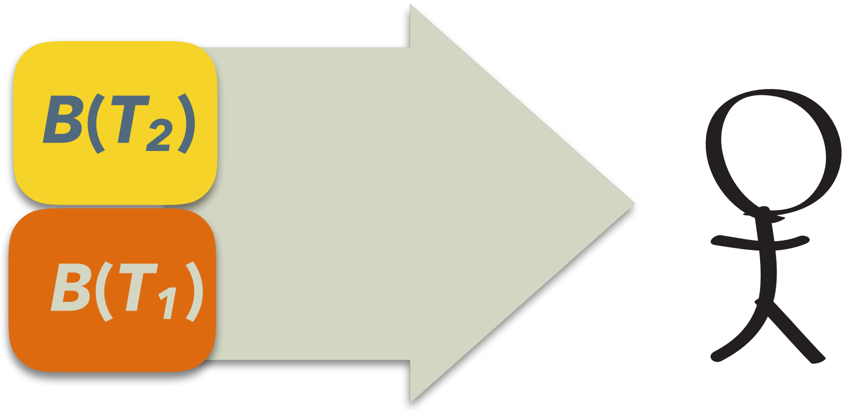
does not happen



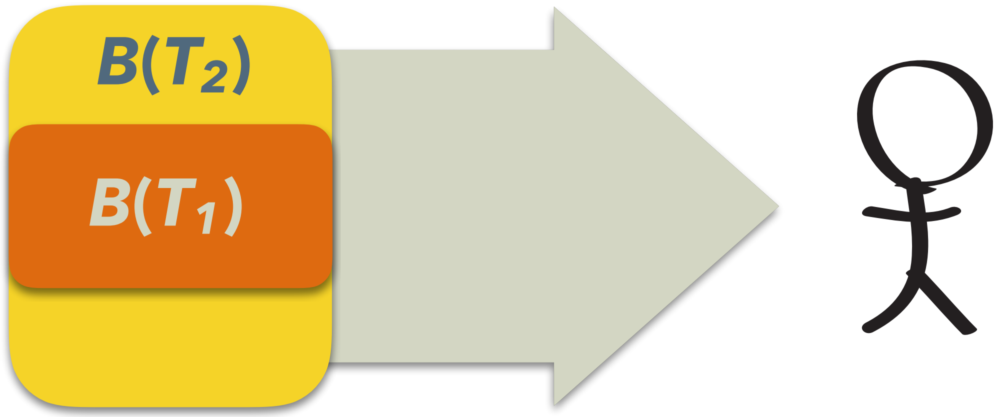
does not happen



does not happen



can happen



can happen

Exercise today

1 Stefan-Boltzmann constant

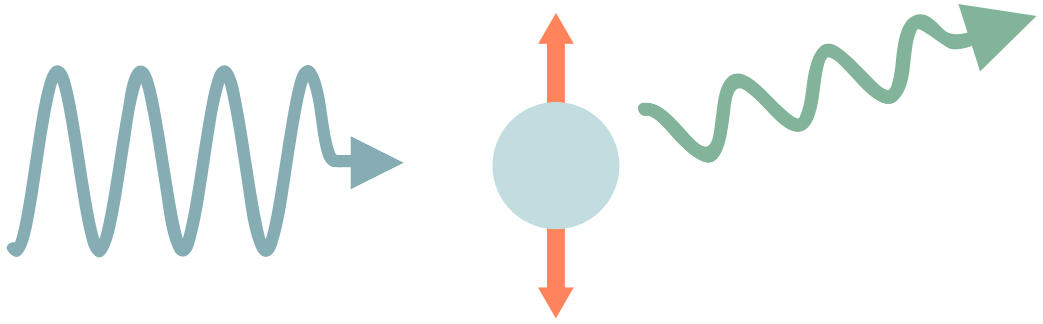
$$\int_0^\infty \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3}$$

Planck function

$$B(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{\exp \frac{h\nu}{kT} - 1}$$

2 Why Thomson scattering is sometims referred to "electron" scattering?



Thomson cross section

$$P = \langle S \rangle \sigma_T$$

$$\sigma_T = \frac{P}{\langle S \rangle} = \frac{8\pi}{3} \frac{e^4}{m^2 c^4}$$

we did not assume it is an electron, but a charge

$$P = \frac{e^4 E_0^2}{3m^2 c^3}$$

Dipole approximation

$$\langle \sin^2 \omega t \rangle = \frac{1}{2}$$

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2 \Theta$$

incoming electric field

$$\mathbf{E} = \boldsymbol{\varepsilon} E_0 \sin \omega t$$

$$m\ddot{\mathbf{r}} = e\mathbf{E}$$

dipole moment

$$\mathbf{d} = e\mathbf{r}$$

$$\ddot{\mathbf{d}} = e\ddot{\mathbf{r}}$$

$$\ddot{\mathbf{d}} = \frac{e^2\mathbf{E}}{m}$$

$$|\ddot{\mathbf{d}}|^2 = \frac{e^4 E_0^2}{2m^2}$$

3 Find

$$\frac{dI}{d\tau} = -I + S$$

$$I = I(0)\, e^{-\tau} + S(1-e^{-\tau})$$

Set $\mathcal{I}_\tau = I_\tau e^\tau$

$$\mathcal{S} = Se^\tau$$

note $\mathcal{S}$ is not dependent on τ

$$\frac{d\mathcal{I}_\tau}{d\tau} = \frac{dI_\tau}{d\tau}e^\tau + I_\tau e^\tau$$

$$\frac{d\mathcal{S}}{d\tau} = Se^\tau$$