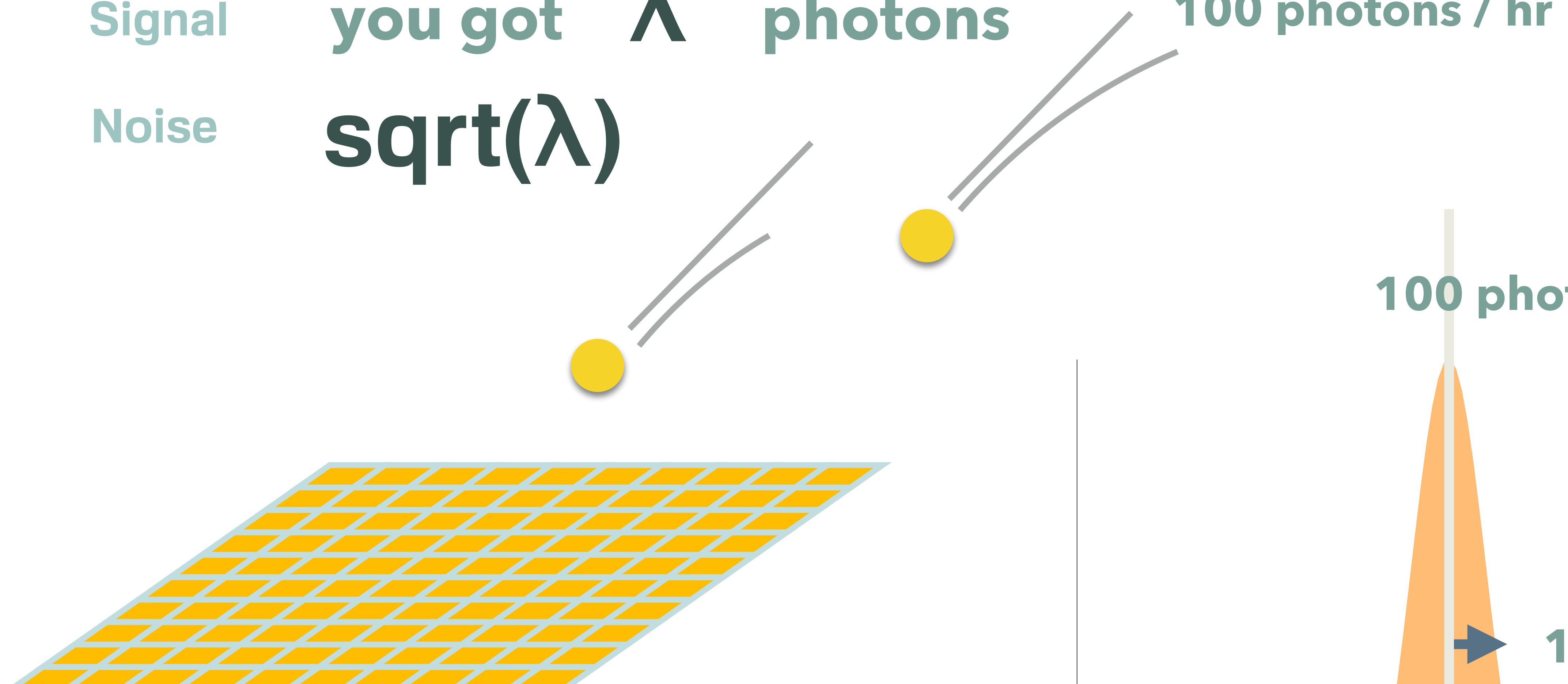
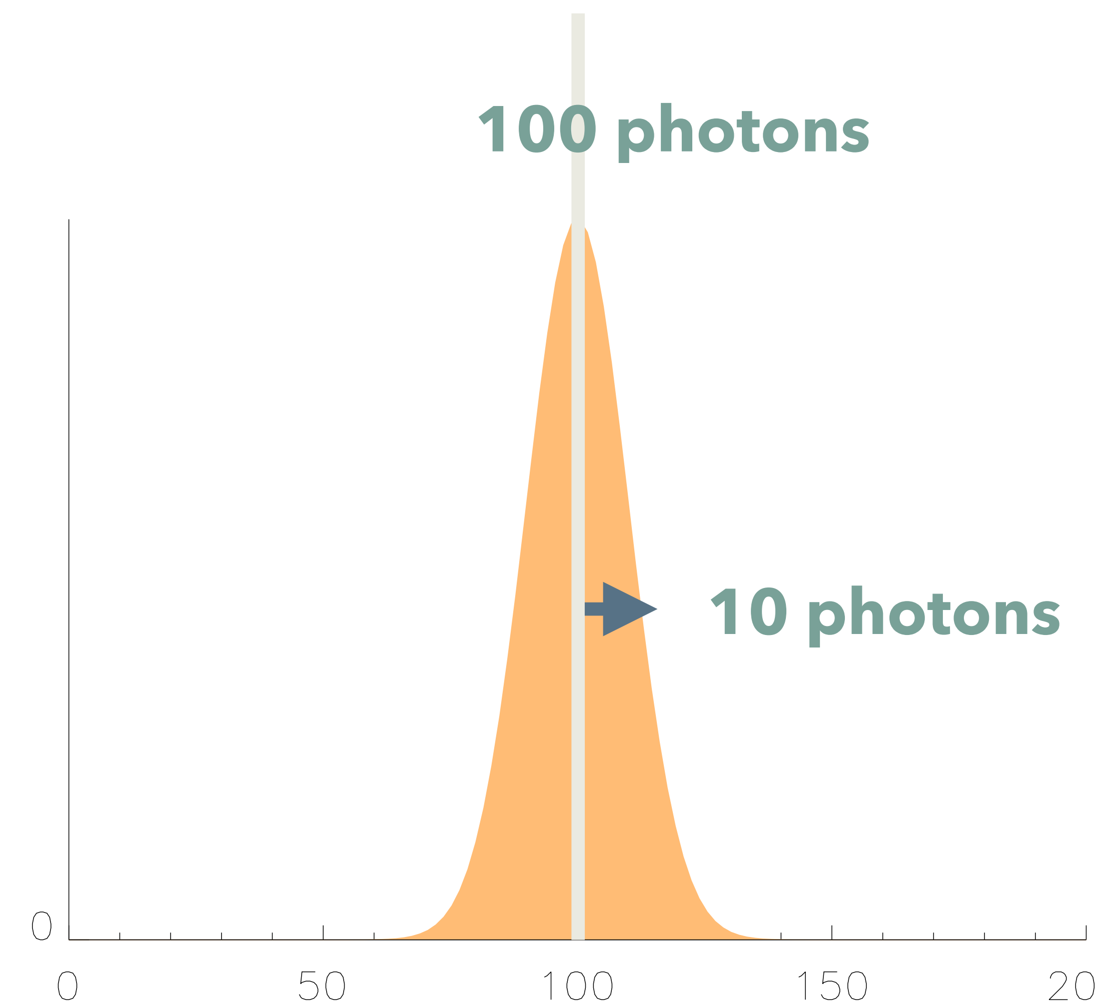
Calculation of signal-to-noise ratio

your photon hitting detector

Signal you got λ photons

Noise sqrt(λ)





when you got 100 photons.
uncertainty is sqrt(100) = 10 photons

when you repeat measurements,
results distribute as a Gaussian function with width sigma = 10 photons

Why sqrt(N)?

1. Binomial distribution

    mean, sigma

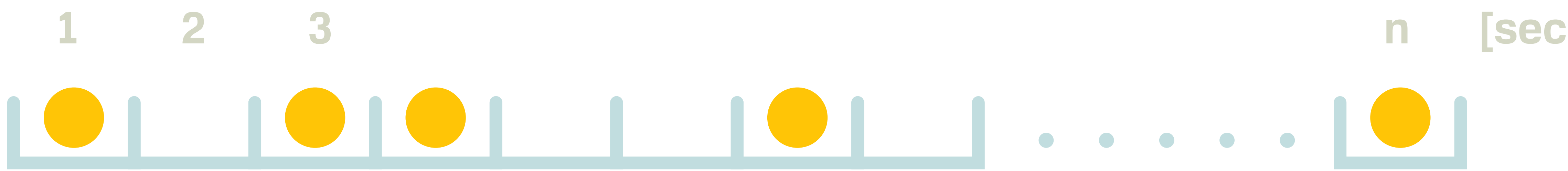
2. Poisson distribution

    $n \rightarrow \infty$

    mean, sigma

3. Gaussian is an approximation of Poisson distribution

    we will skip this



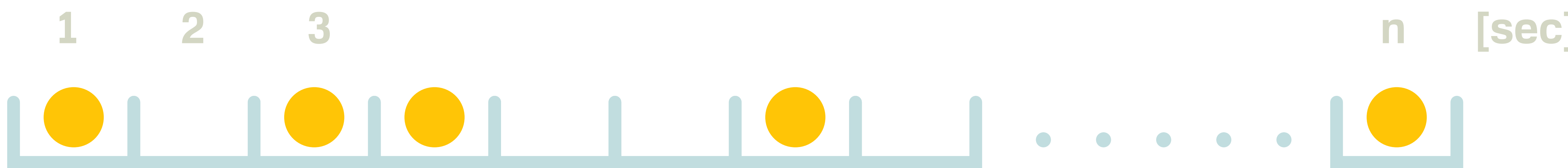
event happens λ times in n seconds on average

like 100 photons / hour

for each box, the probability that you receive a photon is

$$p = \lambda / n$$

if there are more than 1 photon,
we will make box (= unit time)
smaller



event happens λ times in n seconds on average

like 100 photons / hour

for each box, the probability that you receive a photon is

$$p = \lambda / n$$

if there are more than 1 photon, we will make box (= unit time) smaller

Binomial distribution tells probability of

event happens k times

on average happens λ times

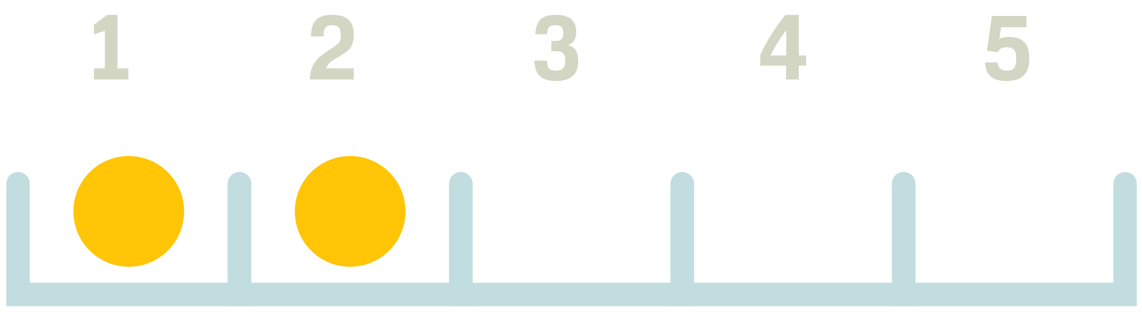
but often observations deviate, like $\lambda - 1$ or $\lambda + 2$ times

how often that happens?

one throw.

a dice.

probability that we get **4** is $p = \frac{1}{6}$



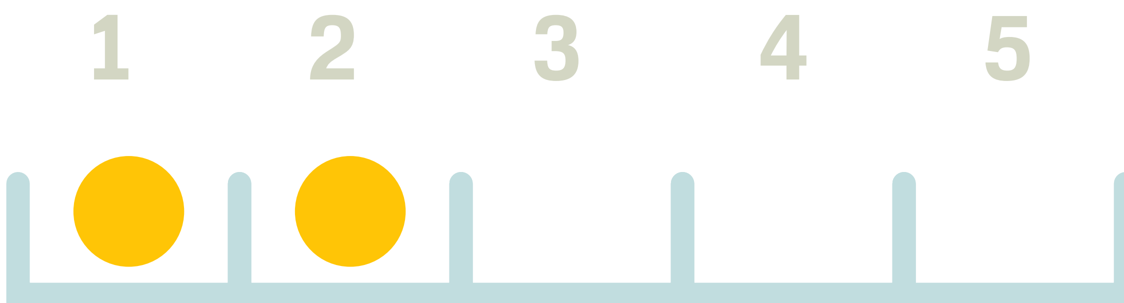
we do not get **4** is

$$q = 1 - p$$

a dice.

one throw.

probability that we get **4** is $p = \frac{1}{6}$



we do not get **4** is

$$q = 1 - p$$

throw a dice. **5** times.

probability that we get **4** in first two boxes 1 and 2 is

$$P = p^2 q^3$$

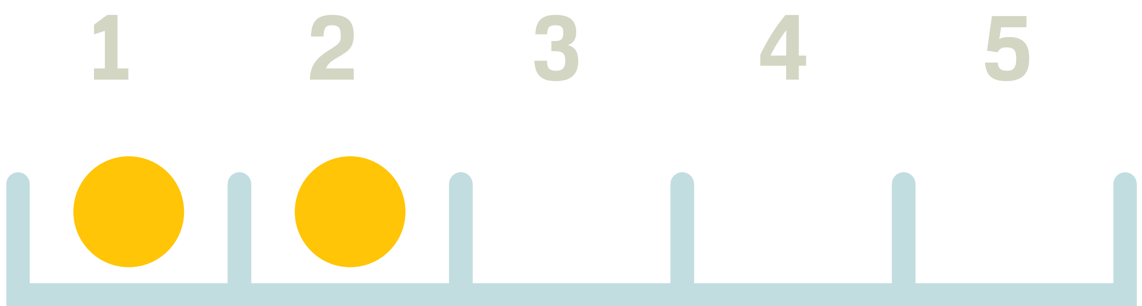
a dice.

one throw.

probability that we get **4** is $p = \frac{1}{6}$

we do not get **4** is

$$q = 1 - p$$



throw a dice. **5** times.

probability that we get **4** in first two boxes 1 and 2 is

$$P = p^2 q^3$$

probability that we get **4** twice any where in **5** boxes

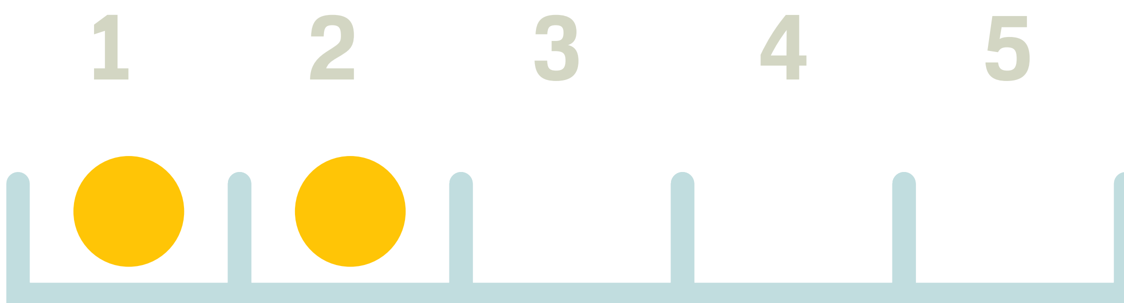
$$P = p^2 q^3$$

× choice of 2 boxes

a dice.

one throw.

probability that we get **4** is $p = \frac{1}{6}$



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$$P = p^2 q^3$$

× choice of 2 boxes

pick **2** boxes out of **5** boxes

pick **x** boxes out of **n** boxes

count all cases $\underbrace{n(n-1)(n-2)\ldots}_{x} = \frac{n!}{(n-x)!}$

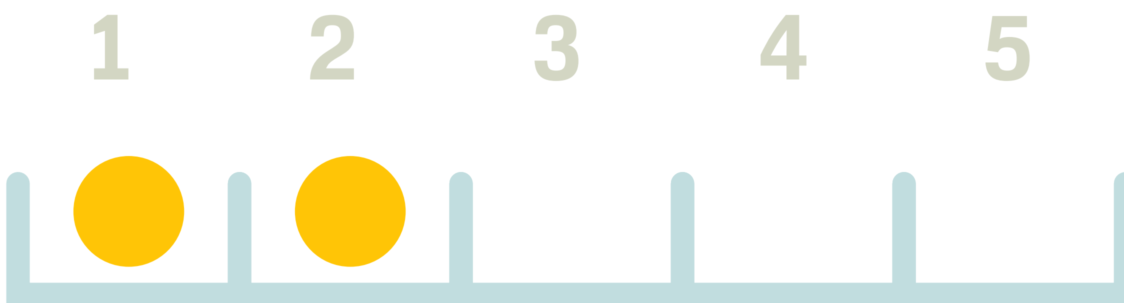
you have **n** choices in first pick

you have **n-1** choices in second pick

a dice.

one throw.

probability that we get **4** is $p = \frac{1}{6}$



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× choice of 2 boxes

pick **2** boxes out of **5** boxes

pick **x** boxes out of **n** boxes

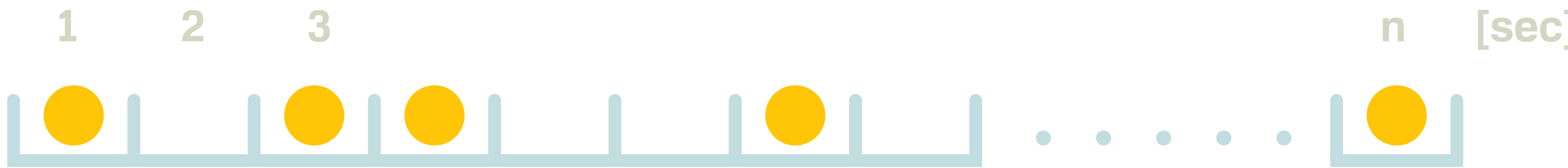
count all cases $\overbrace{n(n-1)(n-2)\ldots}^{x} = \frac{n!}{(n-x)!}$

remove redundancy $x(x-1)(x-2)\ldots = x!$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

you have **n** choices in first pick

you have **n-1** choices in second pick



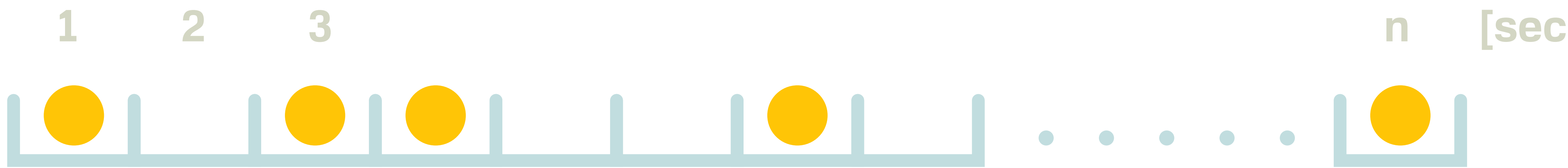
event ● happens **k** times in **n** seconds (on average **λ** times)

$\lambda = np$

$$P(k) = \binom{n}{k} p^k q^{n-k}$$

Binomial distribution

$$P(k) = \frac{n!}{k!\,(n-k)!} p^k q^{n-k}$$



event ● happens **k** times in **n** seconds (times)

(on average **λ** times)

λ = np

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

$$P(x) = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

next tasks : to show

1. average is really **λ** $\qquad \lambda = \sum_{k}^{n} k\, P(k)$ **?**

2. standard deviation is really **sqrt(λ)** $\qquad \sigma^2 = \sum_{k}^{n} (k-\lambda)^2\, P(k)$ **?**

Binomial theorem

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^n = x^n + n \cdot x^{n-1}y + \ldots + n \cdot xy^{n-1} + y^n$$

.....

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

what we want to show

mean

$$\lambda = \sum_{k}^{n} k\, P(k)$$

$$= \sum_{k}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k}^{n} k \frac{n!}{k!\,(n-k)!} p^k q^{n-k}$$

Binomial theorem

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.....

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}$$

what we want to show

mean

$$\lambda = \sum_k k\,P(k)$$

$$= \sum_k^n k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_k^n k \frac{n!}{k!\,(n-k)!} p^k q^{n-k}$$

$$\langle k\rangle = \sum_{k=0}^{n} k\,P(k)$$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

for k=0 term is 0

$$= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

$k' = k-1$

$k' \to n-1$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)![n-1-(k-1)]!} p^{k-1} q^{n-1-(k-1)} \cdot p$$

$$= \sum_{k'=0}^{n-1} \frac{n\cdot(n-1)!}{k'![n-1-k']!} p^{k'} q^{n-1-k'} \cdot p$$

$$= np \sum_{k'=0}^{n-1} \frac{(n-1)!}{k'![n-1-k']!} p^{k'} q^{n-1-k'}$$

Binomial theorem

$$= np(p+q)^{n-1}$$

p+q=1

$$= \lambda$$

λ = np

variance

binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

what we want to show

$$\sigma^2 = \sum_{k=0}^{n} (k-\lambda)^2 \cdot P(k) = \lambda$$

$$\sigma^2 = \sum_{k=0}^{n} (k-\lambda)^2 P(k)$$

$$= \sum_{k=0}^{n} (k^2 - 2\lambda k + \lambda^2) \cdot P(k)$$

$$= \sum_{k=0}^{n} k^2 P(k) - 2\lambda \sum_{k=0}^{n} k P(k) + \lambda^2 \sum_{k=0}^{n} P(k)$$

$$= \sum_{k=0}^{n} k^2 P(k) - \lambda^2$$

$$\sum_{k=0}^{n} k^2 P(k) = \sum_{k=0}^{n} k(k-1)P(k) + \sum_{k=0}^{n} kP(k)$$

$$= \sum_{k=0}^{n} k(k-1)P(k) + \lambda$$

$$= \sum_{k=2}^{n} k(k-1)P(k) + \lambda$$

when

k=0, 1

k(k-1)P(k) =0

variance

binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

what we want to show

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$$= \sum_{k=2}^{n} k(k-1)P(k) + \lambda$$

when

k=0, 1

k(k-1)P(k) =0

variance

$$\sum_{k=2}^{n} k(k-1)P(k) = \sum_{k=2}^{n} k(k-1)\frac{n!}{k!(n-k)!}p^k q^{n-k}$$

$$= \sum_{k=2}^{n} \frac{n(n-1)\cdot(n-2)!}{(k-2)![n-2-(k-2)]!}p^{k-2}q^{n-2-(k-2)}\cdot p^2$$

$$\sum_{k=0}^{n} k^2 P(k) = \sum_{k=2}^{n} k(k-1)P(k) + \lambda$$

$$= n(n-1)p^2(p+q)^{n-2}$$

$k' = k\text{-}2$

$k' : 0 \rightarrow n\text{-}2$

$$= n(n-1)p^2$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\sigma^2 = \sum_{k=0}^{n}(k-\lambda)^2 P(k)$$

$$= \sum_{k=0}^{n}(k^2 - 2\lambda k + \lambda^2)\cdot P(k)$$

$$= \sum_{k=0}^{n} k^2 P(k) - 2\lambda \sum_{k=0}^{n} kP(k) + \lambda^2 \sum_{k=0}^{n} P(k)$$

$$= \sum_{k=0}^{n} k^2 P(k) - \lambda^2$$

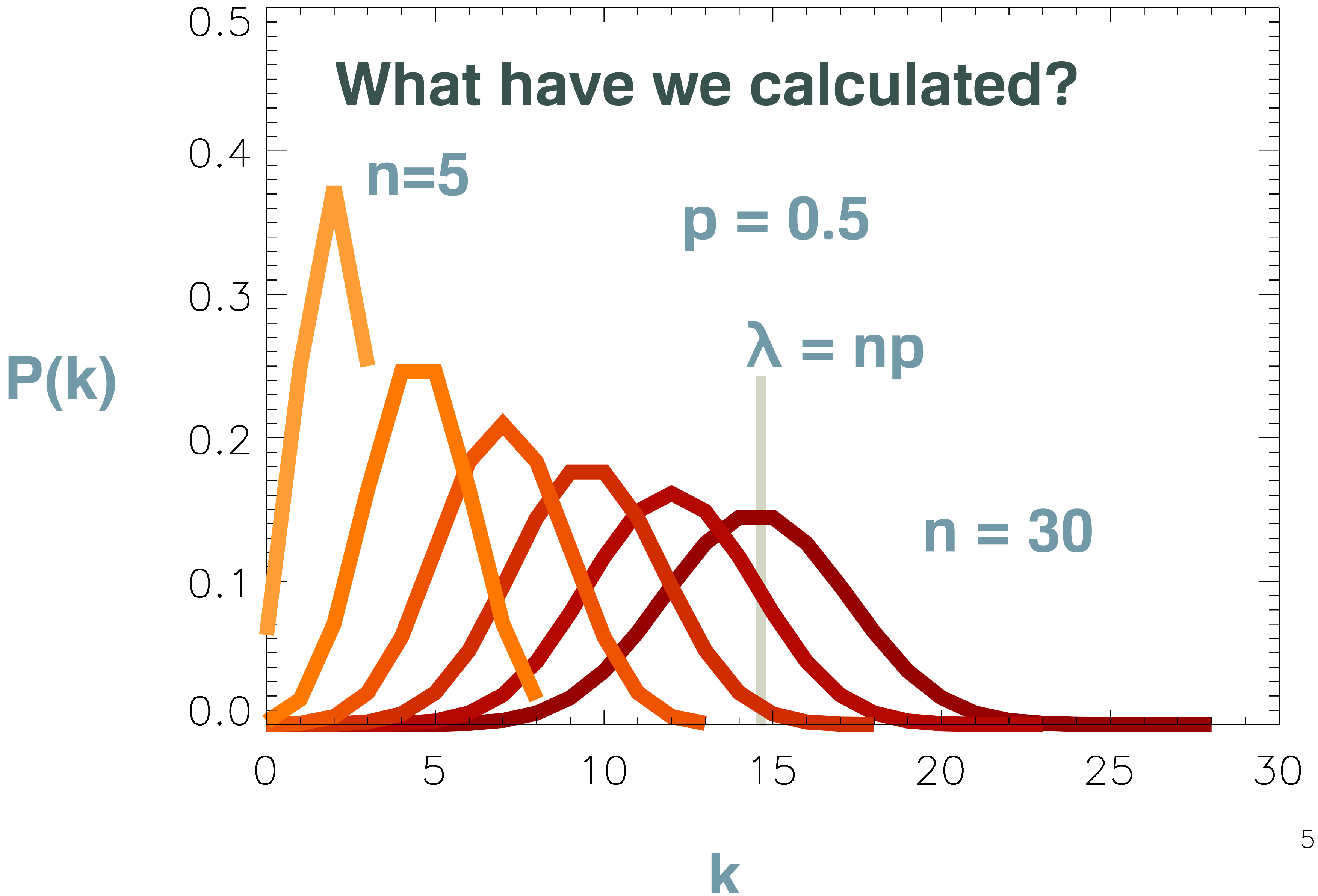
$$\sigma^2 = n(n-1)p^2 + np - (np)^2$$

$$= np(1-p)$$

$$= npq$$

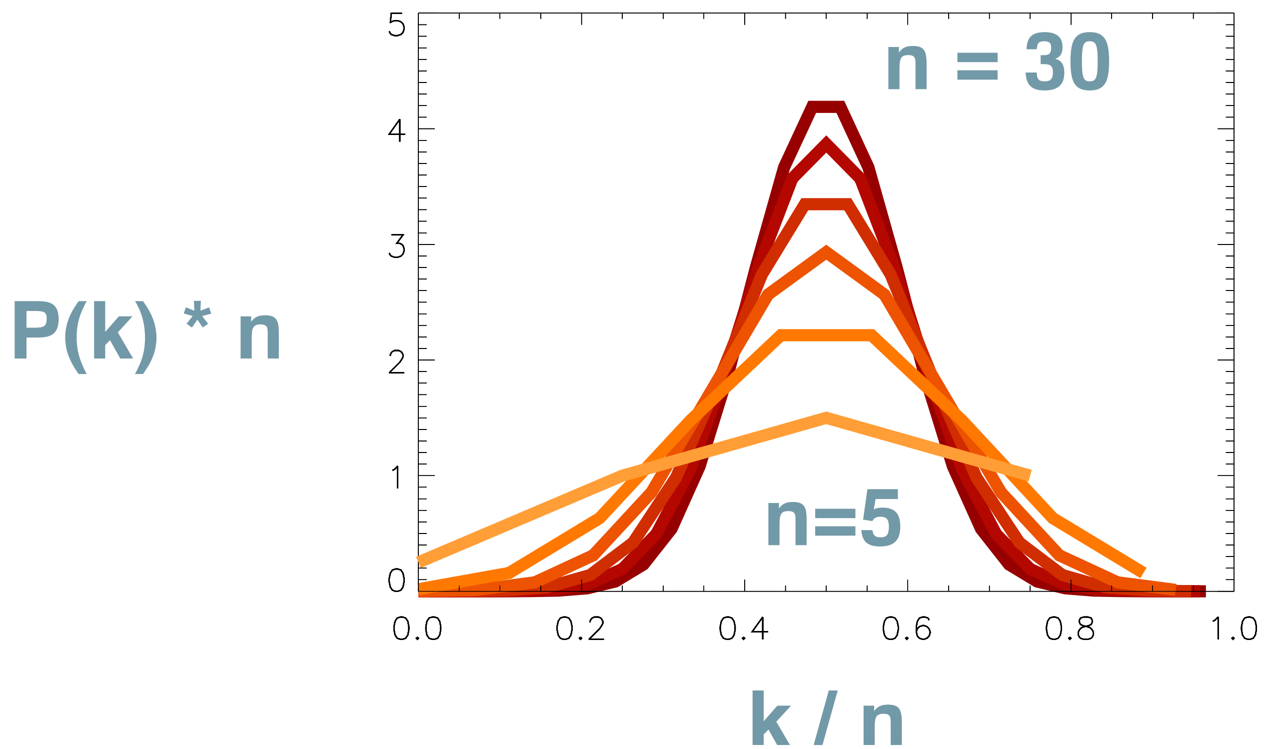
when n→∞ $\sigma^2 = np = \lambda$

What have we calculated?



$$P(k) = \frac{n!}{k!\,(n-k)!}\,p^k q^{n-k}$$

$\sigma^2 = npq$



Binomial → Poisson

$$P(k) = \binom{n}{k} p^k q^{n-k}$$

λ = np

put it front

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

Poisson distribution

Binomial → Poisson   n → ∞

λ = np

$$P(k) = \binom{n}{k} p^k q^{n-k}$$

$$= \frac{n \cdot (n-1) \cdot (n-2) \ldots (n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

put it front

$$= \frac{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-k+1}{n}}{k!} \lambda^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

n → ∞

$$\Rightarrow \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$n \to \infty$

$$\left(1 - \frac{\lambda}{n}\right)^{-k} \to 1$$

$$\Rightarrow \frac{\lambda^k}{k!} \left[1 + \left(-\frac{\lambda}{n}\right)\right]^{\left(-\frac{n}{\lambda}\right) \cdot -\lambda}$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

Poisson distribution

Poisson mean

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\langle k \rangle = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda e^{\lambda}$$

$$= \lambda$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} \ldots.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots.$$

Taylor series

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Poisson variance

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\sum_{k=0}^{\infty} k^2 \, P(k) = \sum_{k=0}^{\infty} k(k-1) \, P(k) + \sum_{k=0}^{\infty} k \, P(k)$$

$$= \sum_{k=0}^{\infty} k(k-1) \, P(k) + \lambda$$

$$\sigma^2 = \sum_{k=0}^{\infty} (k-\lambda)^2 \, P(k)$$

$$= \sum_{k=0}^{\infty} k^2 \, P(k) - \lambda^2$$

Poisson variance

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$$= \sum_{k=0}^{\infty} k(k-1) \, P(k) + \lambda$$

$$\sum_{k=0}^{\infty} k(k-1) \, P(k) = \sum_{k=2}^{\infty} k(k-1) \, P(k)$$

$$= \sum_{k=2}^{\infty} k(k-1) \, \frac{\lambda^k}{k!} e^{-\lambda}$$

when

k=0, 1

k(k-1)P(k) =0

Poisson variance

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\sigma^2 = \sum_{k=0}^{\infty} (k - \lambda)^2 \, P(k)$$

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$$= \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!}$$

$$= \lambda^2 e^{-\lambda} e^{\lambda}$$

$$= \lambda^2$$

Poisson variance

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\sigma^2 = \sum_{k=0}^{\infty} (k-\lambda)^2 \, P(k)$$

$$= \sum_{k=0}^{\infty} k^2 \, P(k) - \lambda^2$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots.$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

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Poisson variance

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

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$$= \sum_{k=0}^{\infty} k(k-1) P(k) + \lambda$$

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$$= \sum_{k=2}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!}$$

$$= \lambda^2 e^{-\lambda} e^{\lambda}$$

$$= \lambda^2$$

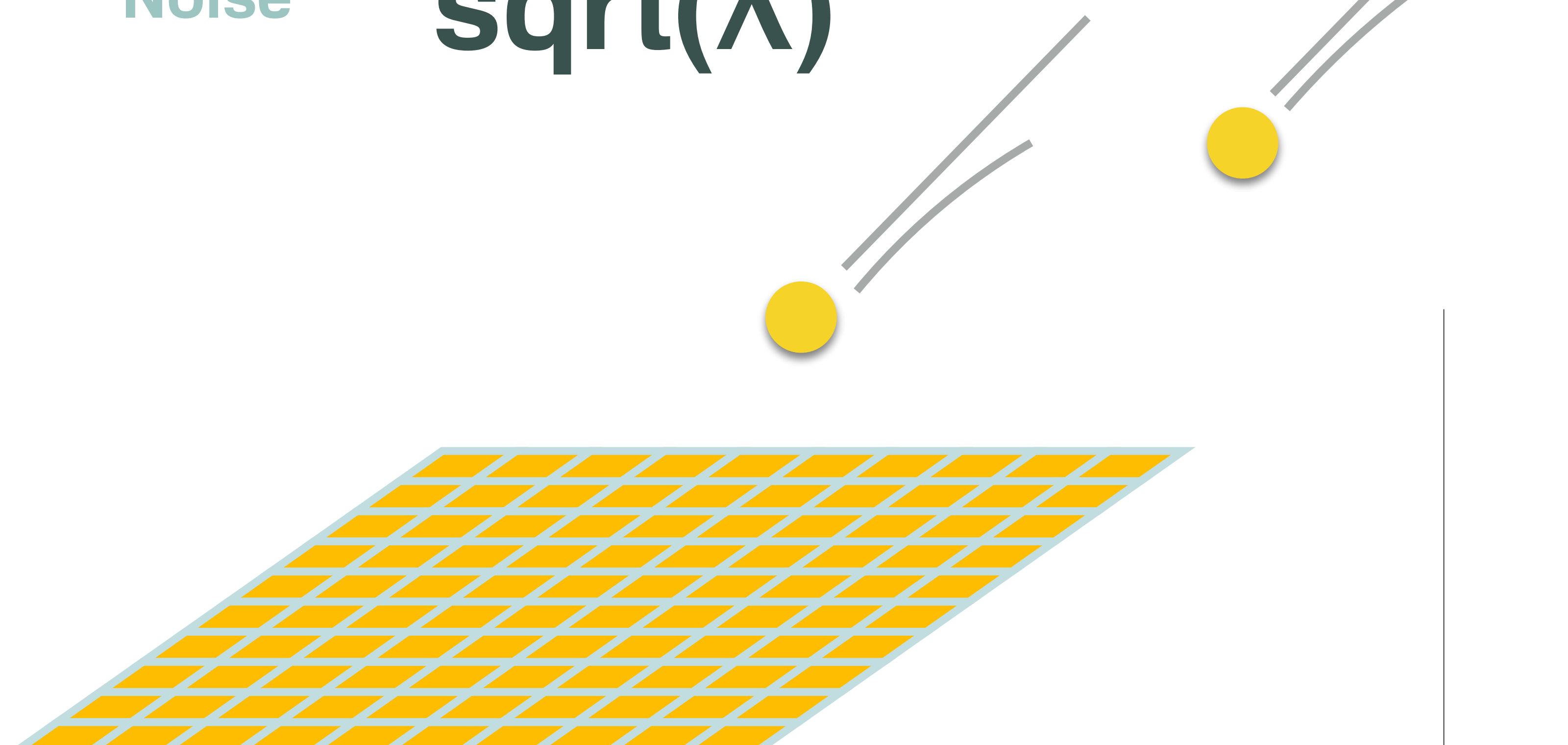
$$\sigma^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

calculation of signal-to-noise ratio

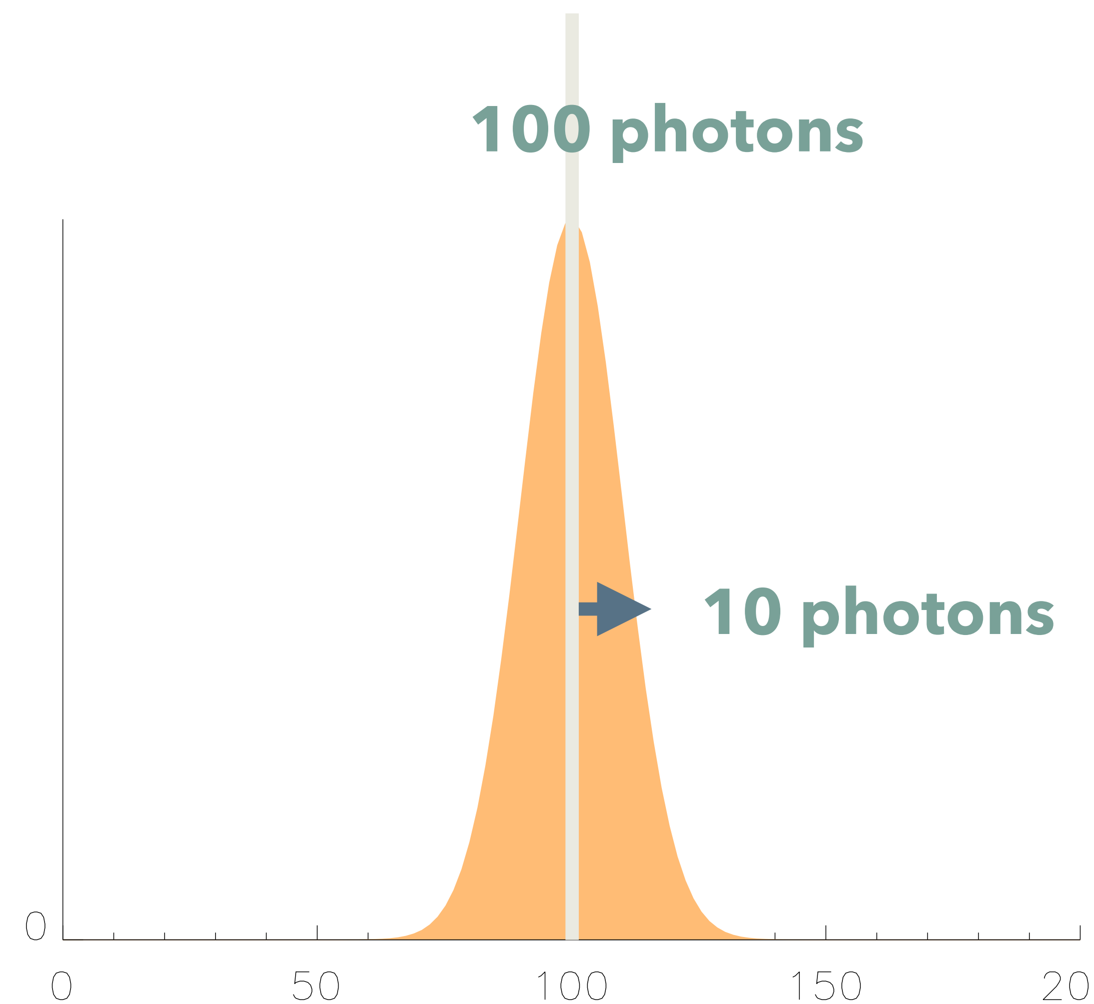
your photon hitting detector

Signal you got λ photons

Noise sqrt(λ)



when you got 100 photons.
uncertainty is sqrt(100) = 10 photons



Calculation of signal-to-noise ratio

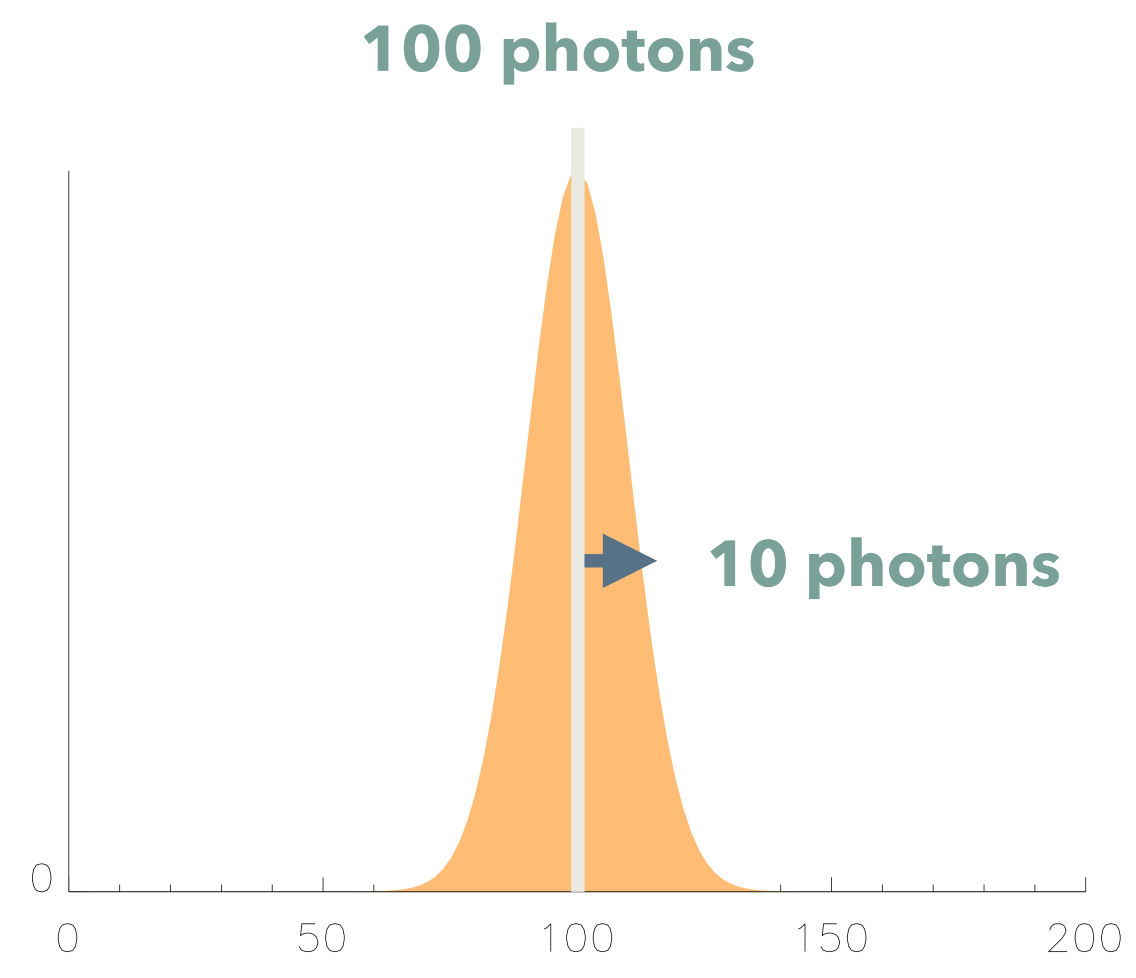
1. 100 photons
2. 100 photons x 100 frames
3. 1000 DN gain 10 e-
4. noise formula what is background-limited performance
5. pixel scale changed
6. diffraction limited
7. larger telescope

1 you got **100 photons**

signal	$S = 100$
variance	$\sigma^2 = 100$
standard deviation	$\sigma = 10$

signal-to-noise ratio

$$S/N = 100/10 = 10$$



2 you got **100 photons** per frame

- you got **100 frames**

signal	$S = 100 \times 100 = 10000$ photons
variance	$\sigma^2 = 10000$
standard deviation	$\sigma = 100$

signal-to-noise ratio

$$S/N = 10000/100 = 100$$

2 you got **100 photons** per frame

- you got **100 frames**

signal	$S = 100 \times 100 = 10000$	photons
variance	$\sigma^2 = 10000$	
standard deviation	$\sigma = 100$	

signal-to-noise ratio

$$S/N = 10000/100 = 100$$

- you got **N frames**

signal	$S = 100 \times N = 100\,N$	photons
variance	$\sigma^2 = 100N$	
standard deviation	$\sigma = 10\sqrt{N}$	

signal-to-noise ratio

$$S/N = 100\,N / (10\sqrt{N}) = 10\,\sqrt{N}$$

S/N only improves with $\sqrt{t}$

3 1000 DN gain 10 e-

DN (digital number) ADU (analog-digital unit)

what follows Poisson statistics is photon not ADU

signal	$S = 1000 * 10 = 10000$	electrons ~ photons
variance	$\sigma^2 = 10000$	
standard deviation	$\sigma = 100$	

signal-to-noise ratio

$$S/N = 10000/100 = 100$$

wrong

signal	$S = 1000$
variance	$\sigma^2 = 1000$
standard deviation	$\sigma = 32$
S/N	$S/\sigma = 32$

4 noise formula — what is background limited performance

converted to photons

signal $= S_{\text{obj}}$

noise $= \sqrt{R_{\text{read}}^2 + S_{\text{bg}} + S_{\text{obj}}}$

readout noise has to be squared

$$\left(= \sqrt{\frac{R_{\text{read}}^2}{N_{\text{DR}}} + (\Phi_{\text{bg}} + \Phi_{\text{obj}})\frac{t}{g_{\text{ain}}}} \right)$$

multiple readout
gain
dark current
quantum efficiency

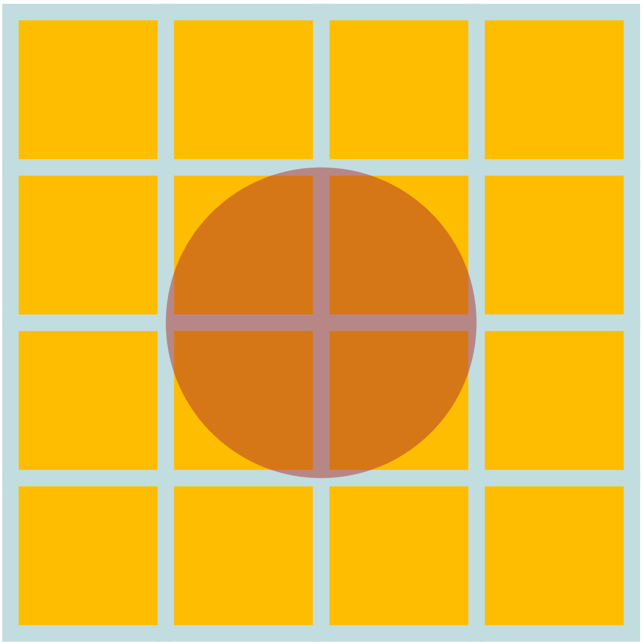
S/N $= \dfrac{S_{\text{obj}}}{\sqrt{R_{\text{read}}^2 + S_{\text{bg}} + S_{\text{obj}}}}$

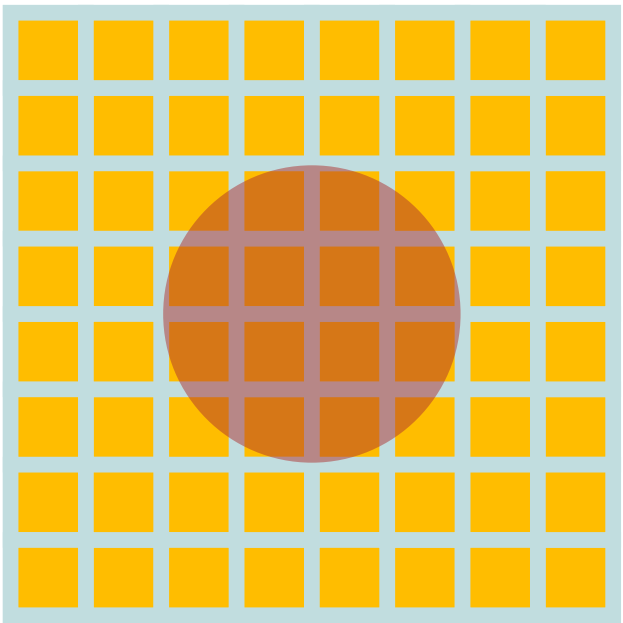
background limited performance

when S_{bg} is dominant

5 different pixel scales

which pixel size you should use?
point source
meet at least Nyquist sampling





photometric S/N

	signal	noise
read out noise limited	same	?
background or source limited	same	?

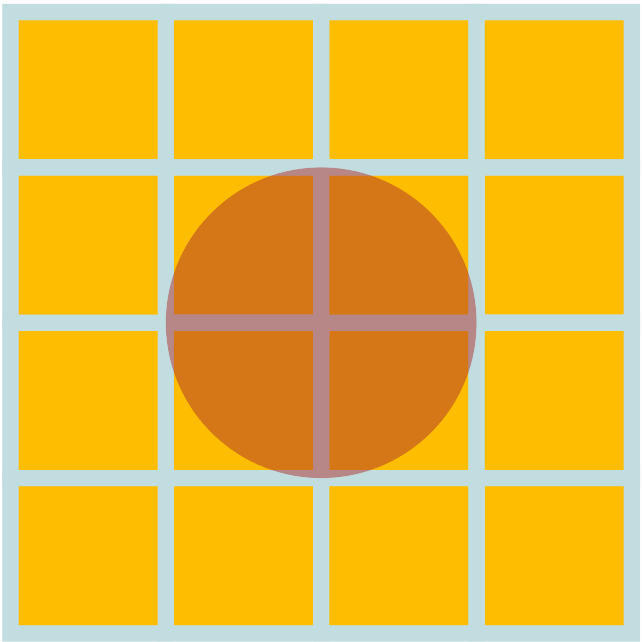
in practice with smaller pixel size, the field of view is smaller

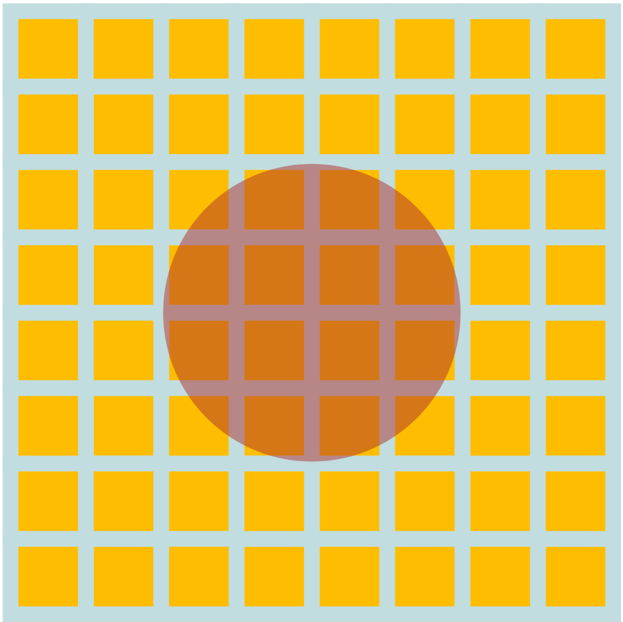
5 different pixel scales

which pixel size you should use?

point source

meet at least Nyquist sampling





photometric S/N

	signal	noise
read out noise limited	same	x 2
background or source limited	same	x 1

in practice with smaller pixel size, the field of view is smaller

6 diffraction limited point source

with AO or without AO? (adaptive optics)

seeing ~1" at all wavelength

improves slightly toward longer λ

0".3 at 3-10 um

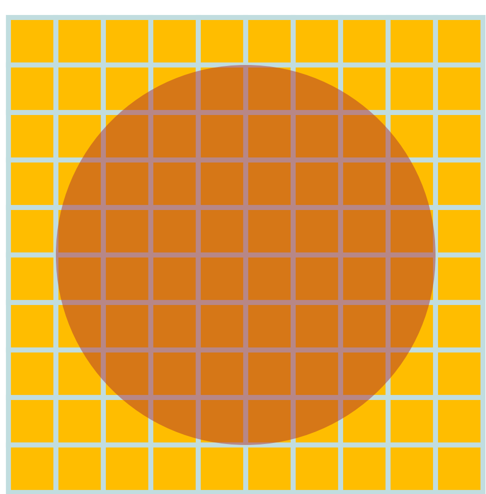
0".5 at 2 um while 1" at V

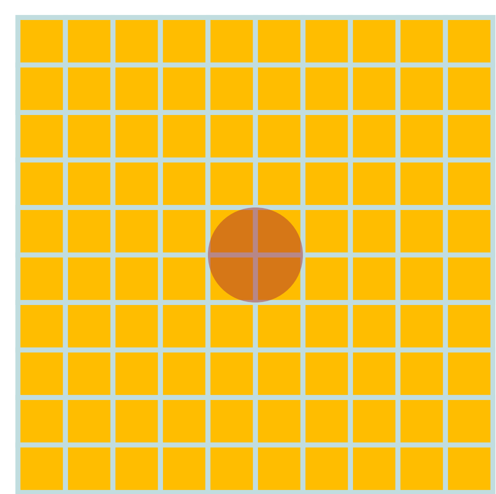
diffraction limited

λ/D

$\lambda = 2\ \mu m$ D = 8 m

50 mas = 0.05"





x 1/10 (0.5" vs 0.05")

$S_{bg} \rightarrow (1/10)^2\ S_{bg}$

if background limited

signal stays same

noise → 1/10

S/N → x10 boost

7 larger telescope 1 seeing limited

seeing stays same ~1”

D^2 photon bucket

$$S_{bg} \rightarrow D^2 \times S_{bg}$$

$$S_{obj} \rightarrow D^2 \times S_{obj}$$

noise $\sqrt{S_{bg} + S_{obj}} \rightarrow D \times \sqrt{S_{bg} + S_{obj}}$

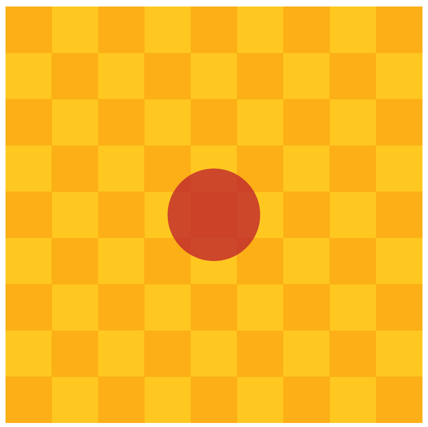
S/N $1 \rightarrow D$

since S/N improves as $\sqrt{t}$

to increase S/N by D with a small telescope
takes D^2 longer integration time

Exercise today

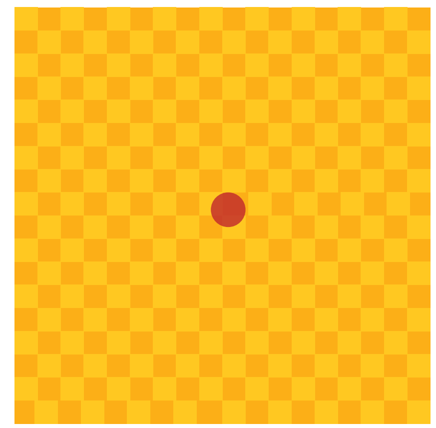
larger telescope **diffraction limited**



VLT

diffraction limited

D	8 m
Signal	1
diff	0″.05



ELT

diffraction limited

D	40 m
Signal	25
diff	0″.01

1. **background limited case**
2. **readout limited case**