Astrophysik II: Galaxien und Kosmologie

WS17/18

Übungsblatt 8

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Aufgabe 1. *The Friedmann Equations*
In Cosmology we utilize the FLRW metric as it can describe the three possible constant curvature geometries of an isentropic homogeneous universe. Under this metric the squared line element takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{1}$$

where t is the time coordinate, r is the radial coordinate and the metric on the two-sphere is $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$ (comoving coordinates), a(t) is the scale parameter and the value of k sets the curvature normalized such that $k \in \{+1, 0, -1\}$. Together with Einsteins field equation;

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}\mathcal{R}\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2}$$

one can obtain equations of motion which describe the evolution of the scale parameter $R(t)$.

(a) Use the equations above to derive the Friedmann equations.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}} \tag{3}$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(c^2\rho + 3P) + \frac{\Lambda}{3}} \tag{4}$$

Where the Christoffel symbols are given by,

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{5}$$

the Riemann tensor by,

$$\mathcal{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{6}$$

the Ricci tensor by,

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^\lambda{}_{\mu\lambda\nu} \tag{7}$$

the Ricci scalar by,

$$\mathcal{R} = \mathcal{R}^\mu{}_\mu = g^{\mu\nu}\mathcal{R}_{\mu\nu} \tag{8}$$

and finally the energy-momentum tensor,

$$T_{\mu\nu} = (P + c^2\rho)\frac{U_\mu U_\nu}{c^2} + Pg_{\mu\nu} \tag{9}$$

where the four-velocity in comoving coordinates is $U^\mu = (1, 0, 0, 0)$.

(b) Use the results of part (a) to derive an expression for conservation of energy with a state parameter $w = P/\rho$. hint: Take the difference between the two Friedmann equations.